You have four hours to complete this exam. When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

**PhD Pass:** Three numbered questions solved completely, or two questions solved completely and substantial progress on two additional questions.

**MS Pass:** Substantial progress on three questions.
Section A

Question 1

Let \( G \) be a simple, undirected, connected graph. For two vertices \( a, b \in V(G) \), we let \( d(a, b) \) denote the distance between them, i.e. the length of a shortest \( ab \)-path in \( G \). We let \( D(a, b) \) denote the length of a longest \( ab \)-path in \( G \).

(a) Prove that, for distinct vertices \( a, b, c \in V(G) \),

\[
d(a, b) + d(b, c) \geq d(a, c).
\]

(b) Prove that, for distinct vertices \( a, b, c \in V(G) \),

\[
D(a, b) + D(b, c) \geq D(a, c).
\]

(c) Prove that \( d(a, b) + d(b, c) = d(a, c) \) if and only if \( b \) lies on a shortest \( ac \)-path.

Question 2

Let the Ramsey number \( R(k, l) \) denote the smallest integer \( n \) such that every red/blue-coloring of \( K_n \) contains either a blue clique on \( k \) vertices or a red clique on \( l \) vertices.

(a) Show that \( R(2, n) = R(n, 2) = n \).

(b) Show that

\[
R(k, l) \leq R(k - 1, l) + R(k, l - 1).
\]

(c) Use the previous results to show that \( R(k, l) \leq \binom{k+l}{k+1} \).

Section B

Question 1

Let \( d_k \) denote the number of length-\( k \) words from the alphabet \( \{0, 1, 2, 3\} \) with an even number of 0s.

(a) Find a recurrence for the \( d_k \).

(b) Find a closed formula for the ordinary generating function \( D(x) = \sum_{k \geq 0} d_k x^k \).

(c) Find a closed formula for \( d_k \) and confirm by direct enumeration that your formula provides the correct answer for \( k = 3 \).
Question 2

A partition \( \lambda \) of \( n \) is a weakly decreasing tuple of positive integers: \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \). Let \( \text{Par}(n) \) denote the set of partitions of a given integer \( n \).

(a) Give the definition of the dominance partial order, \( \preceq \), on \( \text{Par}(n) \).

(b) Draw the Hasse diagram for the dominance order on \( \text{Par}(6) \).

(c) Express, in terms of \( \preceq \), a necessary condition for which monomial symmetric functions \( m_\mu \) can appear in the monomial symmetric function expansion of a given Schur polynomial \( s_\lambda \). (Assume that \( \lambda \) and \( \mu \) are both partitions of the same integer \( n \).) Make sure to justify your answer.

Section C

Question 1

Recall that the \( n \)th standard permutahedron \( P_n \) is the convex hull in \( \mathbb{R}^n \) of set of points \((\sigma(1), \ldots, \sigma(n))\) where \( \sigma \) ranges over all permutations in \( S_n \). The following questions do not require proofs.

(a) Draw \( P_3 \) and give an inequality description of \( P_n \) for all \( n \). Hint: You may take the normal vectors for the facets to be \( 0 - 1 \) vectors.

(b) Express \( P_n \) as a Minkowski sum of line segments. What is the name for a polytope expressible in this way?

(c) What is the volume of \( P_n \)? Hint: Cayley’s formula from graph theory.

Question 2

Let \( G \) be a bipartite graph with bipartition \((S, T)\) of \( V(G) \).

(a) Define a transversal matroid \( M \) associated to \( S \), and prove that the set of partial transversals satisfy the independent set axioms for a matroid. Hint: augmenting paths.

(b) What is \( M \) when \( G \) is a complete bipartite graph?

(c) Give an example of a transversal matroid which is not graphic.