COMBINATORICS QUALIFYING EXAM
August 2021

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass: Three numbered questions solved completely, or two questions solved completely and substantial progress on two additional questions.

MS Pass: Substantial progress on three questions.

Section A

Question 1

Let G be a simple, undirected, connected graph. For two vertices $a, b \in V(G)$, we let $d(a, b)$ denote the distance between them, i.e. the length of a shortest ab-path in G. We let $D(a, b)$ denote the length of a longest ab-path in G.

(a) Prove that, for distinct vertices $a, b, c \in V(G)$,

$$d(a, b) + d(b, c) \geq d(a, c).$$

(b) Prove that, for distinct vertices $a, b, c \in V(G)$,

$$D(a, b) + D(b, c) \geq D(a, c).$$

(c) Prove that $d(a, b) + d(b, c) = d(a, c)$ if and only if b lies on a shortest ac-path.

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Question 2

Let the Ramsey number $R(k, l)$ denote the smallest integer n such that every red/blue-coloring of K_n contains either a blue clique on k vertices or a red clique on l vertices.

(a) Show that $R(2, n) = R(n, 2) = n$.

(b) Show that

$$R(k, l) \leq R(k - 1, l) + R(k, l - 1).$$

(c) Use the previous results to show that $R(k, l) \leq \binom{k+l}{k}$.

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Section B

Question 1

Let d_k denote the number of length-k words from the alphabet $\{0, 1, 2, 3\}$ with an even number of 0s.

(a) Find a recurrence for the d_k.

(b) Find a closed formula for the ordinary generating function $D(x) = \sum_{k \geq 0} d_k x^k$.

(c) Find a closed formula for d_k and confirm by direct enumeration that your formula provides the correct answer for $k = 3$.

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Question 2

A *partition* λ of n is a weakly decreasing tuple of positive integers: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let $\mathrm{Par}(n)$ denote the set of partitions of a given integer n.

(a) Give the definition of the *dominance partial order*, $\trianglelefteq$, on $\mathrm{Par}(n)$.

(b) Draw the Hasse diagram for the dominance order on $\mathrm{Par}(6)$.

(c) Express, in terms of $\trianglelefteq$, a necessary condition for which monomial symmetric functions m_μ can appear in the monomial symmetric function expansion of a given Schur polynomial s_λ. (Assume that λ and μ are both partitions of the same integer n.) Make sure to justify your answer.

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Section C

Question 1

Recall that the nth standard permutahedron P_n is the convex hull in $\mathbb{R}^n$ of set of points $(\sigma(1), ..., \sigma(n))$ where σ ranges over all permutations in S_n. The following questions do not require proofs.

(a) Draw P_3 and give an inequality description of P_n for all n. Hint: You may take the normal vectors for the facets to be $0-1$ vectors.

(b) Express P_n as a Minkowski sum of line segments. What is the name for a polytope expressible in this way?

(c) What is the volume of P_n? Hint: Cayley's formula from graph theory.

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Question 2

Let G be a bipartite graph with biparition (S, T) of $V(G)$.

(a) Define a transversal matroid M associated to S, and prove that the set of partial transversals satisfy the independent set axioms for a matroid. Hint: augmenting paths.

(b) What is M when G is a complete bipartite graph?

(c) Give an example of a transversal matroid which is not graphic.

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