COMBINATORICS QUALIFYING EXAM
January 2020

You have three hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass: Four numbered questions solved completely, with at least one from each section.

MS Pass: Substantial progress on three questions, in any section.

Section A

Question 1

Let G be a forest.

(a) Prove that G has $n-c$ edges, where n is the number of vertices and c the number of connected components of G.

(b) Find the average degree of G.

(c) Let $G_1, G_2, \ldots, G_k$ be connected subgraphs of G. Prove that their intersection is either empty or a tree.

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Question 2

(a) State and prove Hall's Theorem.

(b) Let G be bipartite graph with bipartition $V \cup W$, and maximum degree $\Delta(G) \geq 1$. Let S_V be the set of all vertices $v \in V$ such that $d(v) = \Delta(G)$, and let S_W be defined similarly. Show that G has a matching that saturates S_V and a matching that saturates S_W.

(c) Show that G has a matching that saturates $S_V \cup S_W$.

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Question 3

(a) State Turán's Theorem.

(b) Give an example of a graph G that is edge-maximal without a K_3 subgraph, but not extremal.

(c) Determine the value of $\mathrm{ex}(n, K_{1,3})$ for all $n \in \mathbb{N}$.

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Section B

Question 1

Let f_n denote the n-th Fibonacci number (using the standard convention that $f_0 = 0$ and $f_1 = 1$).

(a) Write down a functional equation for the ordinary generating function for $\{\mathrm{f}_n\}_{n\geq 0}$.

(b) Write down a functional equation for the exponential generating function for $\{f_n\}_{n\geq 0}$.

(c) Solve either functional equation.

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Question 2

(a) State the Robinson-Schensted correspondence.

(b) Apply the correspondence to the permutation $\sigma = [4, 6, 1, 2, 5, 3]$ (written in one-line notation).

(c) Explain how the images of σ and σ^{-1} are related; illustrate by a direct computation applied to σ^{-1}.

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Section B

Question 1

Let M be the matroid with ground set $\{a, b, c, d, e, f\}$ and circuits

$$\{\{a, b, e\}, \{b, c, d\}, \{d, f, e\}, \{a, c, f\}, \{a, e, d, c\}, \{a, b, d, f\}, \{b, c, f, e\}\}.$$

(a) Describe a graphic representation of M, and prove that the matroid is self-dual. Explain why this matroid is irreducible. How many independent sets does this matroid have?

(b) Draw the lattice of flats of M, and apply the Möbius function to calculate the characteristic polynomial $\chi_M(q)$. Evaluate this polynomial at 3, and explain the combinatorial meaning of this value in terms of the graph.

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Question 2

Let P be a polygon.

(a) Prove that every triangulation of P is regular (coherent). How many vertices does the secondary polytope of P have?

(b) Characterize when P is a zonotope. Now, suppose P is a zonotope. How many parallelograms will a fine tiling (paving) of P have? Explain why the simple matroid associated to an octagonal zonotope cannot be graphic.

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