Algebra Qualifying Exam — August 2021

PLEASE DO NOT IDENTIFY YOURSELF ON YOUR WORK. THE PROCTOR WILL ASSIGN YOU A LETTER. PLEASE WRITE THIS LETTER ON THE TOP RIGHT OF EACH PAGE OF YOUR WORK.

Format

The exam contains four sections (Sections A, B, B+, and C).

Each section contains three problems.

Section B+ contains new ring theory topics which we wish to include in future exams but are not part of the current syllabus. Exam takers are free to attempt these problems but can also ignore this section completely. Problems solved in section B+ will count towards your total of solved problems in section B.

Finally, each numbered problem has a certain number of lettered subproblems (Parts (a), (b), (c), etc.).

Instructions

You have four hours to complete this exam.

When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass:

Four numbered problems solved completely; the set of problems solved completely must include one from each of sections A, B (or B+), and C.

Substantial progress on two other problems.

MS Pass:

Nine lettered subproblems solved completely; the set of subproblems solved must include two subsets of three subproblems that are all from the same section (here B and B+ count as the same section).

The set of subproblems solved must include one from each section A, B (or B+), and C.

Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order $525 = 3 \cdot 5^2 \cdot 7$.

   (a) Compute the number n_p of Sylow p-subgroups permitted by Sylow's Theorem for each prime p dividing the order of the group G.

   (b) Show that G is not simple.

   (c) Show that G is solvable. For this question you may not use the Feit-Thompson Theorem or Burnside's Theorem.

2. Let $S = \{1, 2, \ldots, n\}$ be a finite set containing n elements and $\mathcal{P}(S)$ be the power set of S, which is the set of all subsets of S. For example, if $S = \{1, 2, 3\}$, then

   $$\mathcal{P}(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\},$$

   where $\varnothing$ is the empty set.

   Define an equivalence relation $\sim$ on $\mathcal{P}(S)$ by setting $A \sim S - A$ for $A \subseteq S$, where $S - A$ is the complement of A in S.

   Define a binary operation on $\mathcal{P}(S)$ by setting $A \Delta B = (A \cup B) - (A \cap B)$ for $A, B \subseteq S$, where $A \cup B$ is the union of the sets A and B, $A \cap B$ is their intersection. The operation Δ is sometimes called the *symmetric difference* of the two sets, and $A \Delta B$ contains the elements that are in exactly one of A or B (but not both).

   (a) Prove that Δ is a well-defined binary operation on the set of equivalence classes $\mathcal{P}(S)/\sim$.

   (b) Prove that the set of equivalence classes $\mathcal{P}(S)/\sim$ is a group under the binary operation Δ.

   (c) Is $\mathcal{P}(S)$ a group under the binary operation Δ?

3. Let G be the group $G = \langle a, b \mid a^3 = b^4 = bab^{-1}a = 1 \rangle$.

   (a) Show that G is a non-abelian group of order 12.

   (b) Show that G is not isomorphic to A_4.

   (c) Show that G is not isomorphic to D_6, the dihedral group with 12 elements.

Section B

4. Let R be a commutative ring and let M be an R-module. In this problem, P is an ideal of R, and M_P is the localization of M at P. Show that the following are equivalent

   (a) $M = 0$.

   (b) $M_P = 0$ for all prime ideals P of R.

   (c) $M_P = 0$ for all maximal ideals P of R.

5. Let F be a field. Let A be a linear transformation on a finite dimensional F-vector space V. Suppose that $\det(t - A) = (t - 1)(t^2 - 2)$. In what follows we will let M be the $F[t]$-module whose elements are the elements of V and where t acts by A on V.

   (a) If $F = \mathbb{Q}$ what are the possible invariant factors of M?

   (b) If $F = \mathbb{Q}$ what are the possible Smith normal forms of the matrix $t - A \in M_3(\mathbb{Q}[t])$?

   (c) If $F = \mathbb{F}_2$, the finite field with two elements, what are the possible invariant factors of M?

6. In this problem, let A and B be commutative rings with 1.

   (a) Prove that an ideal P of A is a prime ideal if and only if A/P is an integral domain.

   (b) Prove that an ideal $\mathfrak{m}$ of A is a maximal ideal if and only if $A/\mathfrak{m}$ is a field.

   (c) Give an example of a ring homomorphism $\sigma\colon A \to B$ and a maximal ideal $\mathfrak{n} \subset B$ such that $\sigma^{-1}(\mathfrak{n})$ is not a maximal ideal.

Section B+

7. Consider the ring $\mathbb{C}[x, y]$ with its graded lexicographic ordering so $x \succ y$ but, for example, $y^2 \succ x$. Let $f_1 = x^2 + xy + x + 1$, $f_2 = xy + y^2 + y - 1$ and $f_3 = x + y$.

   (a) Find a Groebner basis for the ideal $I = (f_1, f_2, f_3)$ with respect to this term order.

   (b) Find a reduced Groebner basis for I with respect to this term order.

   (c) Find a minimal Groebner basis for I with respect to this term order.

   (d) Enumerate all the points of the set $\{(a, b) \in \mathbb{C}^2 \colon f_1(a, b) = f_2(a, b) = f_3(a, b) = 0\}$ (this is a finite set).

8. Let A be a commutative Noetherian ring with 1.

   (a) If P is a prime ideal of A show that $\sqrt{P^n} = P$.

   (b) If $\mathfrak{m}$ is a maximal ideal of A show that $\mathfrak{m}^n$ is primary for all $n \geq 1$.

   (c) Let $A = \mathbb{C}[x, y]$. Consider the ideal $I = (x^2, xy)$. Find a primary decomposition of I (and prove it is a primary decomposition).

   (d) What are the associated primes of I?

9. (a) State and prove Nakayama's Lemma for local rings.

   (b) Let $(R, \mathfrak{m})$ be a local Noetherian ring and let $k = R/\mathfrak{m}$ denote its residue field. Let V be a finitely generated R-module. Let $v_1, v_2, \ldots, v_n \in V$ and let $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$ denote their images in $\overline{V} = V/\mathfrak{m}V$. Show that if $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$ are a basis for $\overline{V}$ as a k-vector space then $v_1, v_2, \ldots, v_n$ generate V as an R-module.

Section C

10. Let $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$.

    (a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.

    (b) Let F be the splitting field of $f(x)$ over $\mathbb{Q}$. Show that if $\alpha \in F$ with $f(\alpha) = 0$ and $\beta = 2 - \alpha^2$, then we also have $f(\beta) = 0$.

    (c) Let $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ in $F[x]$. Show that $\alpha\beta\gamma = 1$.

    (d) Conclude that $\mathbb{Q}(\alpha)$ is Galois with Galois group C_3.

11. Let $\mathbb{Q}(\zeta_{18})$ the field generated by a primitive 18th root of unity over $\mathbb{Q}$.

    (a) How many quadratic extensions of $\mathbb{Q}$ are there in $\mathbb{Q}(\zeta_{18})$?

    (b) For each quadratic extension $K \subset \mathbb{Q}(\zeta_{18})$ there exists some $d \in \mathbb{Z}$ which is square free and for which $K = \mathbb{Q}(\sqrt{d})$. List all squarefree values of $d \in \mathbb{Z}$ that generate a quadratic subfield of $\mathbb{Q}(\zeta_{18})$ as above.

12. Let $\mathbb{F}_3$ be the field with three elements and let $\overline{\mathbb{F}}_3$ be an algebraic closure of $\mathbb{F}_3$. For this problem, let $g(x) = x^2 + x + 2 \in \mathbb{F}_3[x]$ and let $\alpha \in \overline{\mathbb{F}}_3$ satisfy $g(\alpha) = 0$. Finally, let $k = \mathbb{F}_3(\alpha)$.

    (a) Please give a complete list of all monic irreducible polynomials of degree 2 over $\mathbb{F}_3$.

    (b) How many elements β generate the field k over $\mathbb{F}_3$? To put this in other words, how many elements $\beta \in k$ are there such that $k = \mathbb{F}_3(\beta)$?

    (c) Explicitly determine all of the elements $\beta \in k$ that generate the field k over $\mathbb{F}_3$. Please write them in terms of α.

    (d) How many generators does the group $k^\times$ have?

    (e) Explicitly determine all of the generators of $k^\times$. Please write them in terms of α.