Algebra Qualifying Exam — January 2020

PLEASE DO NOT IDENTIFY YOURSELF ON YOUR WORK. THE PROCTOR WILL ASSIGN YOU A LETTER. PLEASE WRITE THIS LETTER ON THE TOP RIGHT OF EACH PAGE OF YOUR WORK.

Format

The exam contains three sections (Sections A, B, and C).

Each section contains three numbered problems (Problems 1, 2, and 3).

Finally, each numbered problem has a certain number of lettered subproblems (Parts (a), (b), (c), etc.).

Instructions

You have three hours to complete this exam.

When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass:

Four numbered problems solved completely.

The set of problems solved completely must include one from each of sections A,B, and C.

Substantial progress on two other problems.

MS Pass:

Nine lettered subproblems solved completely.

The set of subproblems solved must include two subsets of three subproblems that are all from the same section.

The set of subproblems solved must include one from each sections A, B, and C.

Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order 495 (note that $495 = 3^2 \cdot 5 \cdot 11$).

   (a) Show that G has either a normal Sylow 5-subgroup or a normal Sylow 11-subgroup.

   (b) Show that G has a normal subgroup of order 55.

   (c) If a Sylow 5-subgroup of G is normal, show that G has a cyclic subgroup of order 55.

2. Let $\mathcal{G}$ be a graph with a finite number of vertices and edges. Assume $\mathcal{G}$ has two connected components, $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $A = \mathrm{Aut}(\mathcal{G})$ be the group of all automorphisms of $\mathcal{G}$. (An *automorphism* of a graph is any permutation of vertices that sends edges to edges.)

   (a) Prove that if $\mathcal{G}_1$ has a different number of vertices from $\mathcal{G}_2$, then $A \cong \mathrm{Aut}(\mathcal{G}_1) \times \mathrm{Aut}(\mathcal{G}_2)$.

   (b) Give an example to show that the conclusion of (a) may be false if the hypothesis that $\mathcal{G}_1$ and $\mathcal{G}_2$ have different numbers of vertices is removed.

   (c) Describe why A must be solvable if both $\mathcal{G}_1$ and $\mathcal{G}_2$ each have at most 4 vertices.

3. Let G be a finite group, let N be a normal subgroup of G, and let H be any subgroup of G.

   (a) Prove that if the index of N in G is relatively prime to the order of H, then $H \subseteq N$.

   (b) Prove that if H is any Sylow p-subgroup of G for some prime p, then $H \cap N$ is a Sylow p-subgroup of N.

Section B

4. Let R be a ring with 1 and let M be a left R-module. The *annihilator* of M is

$$\operatorname{ann}_R(M) = \{r \in R : rm = 0, \text{ for all } m \in M\}.$$

   (a) Prove that the annihilator of M is a 2-sided ideal of R (do not assume R is commutative).

   (b) Let $R = \mathbb{Z}$ and let M be a finite abelian group (hence M is a $\mathbb{Z}$-module). Prove that the annihilator of M is nonzero.

   (c) Let $m \in M$, we define the *left annihilator* of m to be

$$\operatorname{ann}_R(m) = \{r \in R : rm = 0\}.$$

   This is a left ideal. View R as a left R-module. Let $r_0 \in R$. Show the following isomorphism of left R-modules:

$$Rr_0 \cong R/\operatorname{ann}_R(r_0).$$

5. Let $R = \mathbb{Z}[2i] = \{a + 2bi : a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1} \in \mathbb{C}$.

   (a) Show that the element $2i$ is irreducible in R.

   (b) Show that the ideal $(2i)$ is not a prime ideal in R.

   (c) Explain why R is not a Unique Factorization Domain.

6. Let R and S be commutative rings with 1 and let $\phi\colon R \to S$ be a ring homomorphism that sends 1 to 1.

   (a) Show that if P is a prime ideal of S, then $\phi^{-1}(P)$, its preimage in R under ϕ, is a prime ideal of R.

   (b) Give an example to show that the preimage of a maximal ideal of S need not be a maximal ideal of R.

Section C

7. Let $\zeta = (-1+\sqrt{-3})/2$ be a primitive cube root of 1 in $\mathbb{C}$ and let $F = \mathbb{Q}(\zeta)$. Let $K_2 = F(\sqrt[3]{2})$ and $K_3 = F(\sqrt[3]{3})$ (where you may assume $\sqrt[3]{2}$ and $\sqrt[3]{3}$ are contained in $\mathbb{R}$).

   (a) Explain why both K_2 and K_3 are Galois extensions of F with cyclic Galois groups of order 3.

   (b) Show that $K_2 \neq K_3$. (Hint: One way to do this is if $K_2 = K_3$, find cube roots that are fixed by all elements of the Galois group.)

   (c) Explain why K_2K_3 is Galois over F with abelian Galois group.

8. Let $\mathbb{F}_3$ denote the field of 3 elements and let $p(x) = x^3 + x^2 - 1 \in \mathbb{F}_3[x]$. Let α be a root of $p(x)$ in some algebraic closure of $\mathbb{F}_3$.

   (a) Prove that $p(x)$ is irreducible in $\mathbb{F}_3[x]$.

   (b) Describe the general form of an element of $\mathbb{F}_3(\alpha)$ in terms of $\mathbb{F}_3$ and α.

   (c) Explain why either α or $-\alpha$ generates the multiplicative group of nonzero elements, $\mathbb{F}_3(\alpha)^\times$.

   (d) Is $\mathbb{F}_3(\alpha)$ Galois over $\mathbb{F}_3$? If so, find the Galois group of this extension. (Justify your answers.)

9. Let V be an 8-dimensional vector space over the field F with basis $v_1, v_2, \ldots, v_8$. Let $\phi \in \mathrm{Hom}_F(V, V)$ be defined by

$$\phi(v_i) = v_{i+1} \quad \text{for } 1 \leq i \leq 7 \quad \text{and } \phi(v_8) = v_1$$

   (so ϕ cyclically permutes the basis vectors).

   (a) Assume $F = \mathbb{Q}$. Find the characteristic and minimal polynomial of ϕ, and the Rational Canonical form of ϕ.

   (b) Assume $F = \mathbb{C}$. Find all elementary divisors of ϕ, and find the Jordan Canonical form of ϕ.