ALGEBRA PH.D. QUALIFYING EXAM

October 16, 2010

A passing paper consists of four problems solved completely plus significant progress on two other problems; moreover, the set of problems solved completely must include one from each of Sections A, B and C.

Section A.

In this section you may quote without proof basic theorems and classifications from group theory, group actions, solvable groups, commutators, etc. as long as you state what facts you are using.

1. Assume G is a simple group of order 4851 (note that $4851 = 3^2 \cdot 7^2 \cdot 11$).

(a) Compute the number, n_p, of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3$, 7, and 11; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.

(b) Show that there are distinct Sylow 7-subgroups P and Q such that $|P \cap Q| = 7$.

(c) For P and Q as in (b), let $H = P \cap Q$. Explain briefly why 11 does not divide $|N_G(H)|$.

(d) Show that there is no simple group of this order.
[Hint: How many Sylow 7-subgroups does $N_G(H)$ contain, and is this permissible by Sylow?]

2. Find all groups G, finite or infinite, such that $|\operatorname{Aut}(G)| = 1$.
[You may wish to start with G finite; then observe that appropriate arguments generalize directly to infinte groups too.]

3. Let G be a finite group with the property that the centralizer of every nonidentity element is an *abelian* subgroup of G (such G is called a CA-group).

(a) Prove that every Sylow p-subgroup of G is abelian, for every prime p.

(b) Prove that if P and Q are distinct Sylow subgroups of G, then $P \cap Q = 1$.

Section B.

4. Let R be the ring of all continuous real valued functions on the closed interval $[0, 1]$ (under the usual pointwise addition and multiplication of functions). Let $M = \{f \in R \mid f(1/2) = 0\}$.

(a) Prove that M is a prime ideal and identify the quotient ring (as a well-known ring).

(b) Prove that M is not a principal ideal.

(c) Exhibit an infinite properly increasing chain of ideals of R:

$$I_1 \subset I_2 \subset I_3 \subset \cdots \qquad \text{and let} \qquad I = \bigcup_{i=1}^{\infty} I_i$$

(where you need not reprove that I is an ideal). Explain why I could not be finitely generated. [Hint: One way is to consider ideals of functions that vanish on certain sets.]

5. Let x and y be independent indeterminates over the field $\mathbb{C}$ of complex numbers, and let $R = \mathbb{C}[x, y]/(x^2 - y\,,\, y^2 - x)$.

(a) Explain why R is a finite dimensional vector space over $\mathbb{C}$, and find its dimension.

(b) Prove that R is isomorphic to $\mathbb{C}[x]/(x^4 - x)$.

(c) Show that R is (ring) isomorphic to the direct product of four copies of $\mathbb{C}$.

6. Let t be an indeterminate over $\mathbb{Q}$. Classify all finitely generated modules over the ring $\mathbb{Q}[t]/(t^9)$.

7. (a) Find all possible rational canonical forms for a 4×4 matrix A over $\mathbb{Q}$ that satisfies $A^6 = I$ (where I is the identity matrix).

(b) Find all possible rational canonical forms for a 4×4 matrix A over $\mathbb{F}_2$ that satisfies $A^6 = I$.

Section C.

8. Let K be the splitting field of $(x^2 - 3)(x^3 - 5)$ over $\mathbb{Q}$.

(a) Find the degree of K over $\mathbb{Q}$.

(b) Find the isomorphism type of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

(c) Find, with justification, all subfields F of K such that $[F : \mathbb{Q}] = 2$.

9. Let F be a field of characteristic zero and suppose $F[x]$ contains a polynomial $f(x)$ of degree 6 whose roots are not expressible by radicals over F. Let E be a splitting field of f over F. Prove that $[E : F]$ is divisible by 10.
(State clearly what facts you are quoting from either group theory or field theory. Do not assume f is irreducible.)

10. Let K be a field with 625 elements.

(a) How many elements of K are primitive (field) generators for the extension $K/\mathbb{F}_5$? (Justify)

(b) How many nonzero elements are generators of the multiplicative group $K^\times$? (Justify)

(c) How many nonzero elements of K satisfy $x^{75} = x$? (Justify)

(d) Let F be the subfield of K with 25 elements. How many elements a in F are there such that $K = F(\sqrt{a}\,)$? (Justify)

Alternate Questions

3. Show that Z_3 and S_3 are the only finite groups with exactly three conjugacy classes. [You may quote without proof the fact that Z_2 is the only finite group with exactly two conjugacy classes.] (too easy?)

3. Find all finite groups G such that $\text{Aut}(G)$ is cyclic.

2. If both A and B are abelian and $A \cap B = 1$, prove that AB is abelian.

(c) If both A and B are solvable, prove that AB is solvable.

4. Let R be a commutative ring with 1.

(a) Prove that if R has only finitely many elements, then every nonzero element of R is either a unit or a zero divisor.

6. Describe all irreducible (simple) R-modules where $R = \mathbb{Z}[x]$ is the integer polynomial ring in the variable x.

10. Suppose there were a subfield, F, of $\mathbb{C}$ such that $[\mathbb{C} : F] = 5$. You may assume that $\mathbb{C}$ is algebraically closed to do the following:

(a) Show that F contains a primitive 5^{th} root of unity.

(b) Explain why $\mathbb{C} = F(\sqrt[5]{\alpha})$ for some $\alpha \in F$? (You may quote results about solvable extensions.)