ALGEBRA PH.D. QUALIFYING EXAM

January 12, 2018

Three hours

A passing paper consists of five problems solved completely, or four solved completely plus significant progress on two other problems; in both cases the set of problems solved completely must include one from each of Sections A, B and C.

Section A.

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order 6545 (note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$).

(a) Compute the number, n_p, of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 5$ and 17 (only).

(b) Let P_5 be a Sylow 5-subgroup of G. Prove that if P_5 is not normal in G, then $N_G(P_5)$ has a normal Sylow 17-subgroup. [Keep in mind that $P_5 \trianglelefteq N_G(P_5)$.]

(c) Deduce from (b) and (a) that G has a normal Sylow p-subgroup for either $p = 5$ or $p = 17$.

(d) Deduce from (c) that $Z(G) \neq 1$.

2. Let p be a prime and let P be a non-abelian group of order p^3.

(a) Prove that the center of P has order p, i.e., $|\, Z(P)\,| = p$.

(b) Prove that the center of P equals the commutator subgroup of P, i.e., $Z(P) = P'$.

3. Let G be a finite group of order n and let $\pi : G \to S_n$ be the (left) regular representation of G into the symmetric group of degree n.

(a) Prove that if G has an element x of order 2, then $\pi(x)$ is the product of $n/2$ transpositions.

(b) Prove that if $n = 2m$ where m is odd, then G has a normal subgroup of index 2.
[Hint: Use Cauchy's Theorem and part (a).]

Section B.

4. Let R be a Euclidean Domain with respect to the norm N. Let

$$d = \min\{N(x) \mid x \in R - \{0\}\}.$$

(a) Prove that every nonzero element of norm d is a unit in R.

(b) Let $R = \mathbb{Z}[\sqrt{2}]$. You may assume R is a Euclidean Domain with respect to the norm

$$N(a + b\sqrt{2}) = |\, a^2 - 2b^2\,|.$$

Prove that R contains infinitely many units. [Hint: Find a unit of infinite order.]

5. Let R be a nonzero ring with 1 and let M be an R-module containing an R-submodule N.

(a) Prove that if N and M/N are both finitely generated, then M is also finitely generated.

(b) Assume that M is the direct sum

$$M = \bigoplus_{n=1}^{\infty} M_n \qquad \text{where each } M_n \cong R.$$

Prove that M is not a finitely generated R-module. (Recall that the above direct sum is the set of tuples $(r_1, r_2, r_3, \dots)$ where only finitely many $r_n \neq 0$ in each tuple.)

6. Find the number of similarity classes of 10×10 matrices A with entries from $\mathbb{Q}$ satisfying $A^{10} = I$ but $A^i \neq I$ for $1 \leq i \leq 9$, where I is the identity matrix. (You do not need to exhibit representatives of the classes.)

Section C.

7. Let $f(x) = x^4 - 8x^2 - 1 \in \mathbb{Q}[x]$, let α be the real positive root of $f(x)$, let β be a nonreal root of $f(x)$ in $\mathbb{C}$, and let K be the splitting field of $f(x)$ in $\mathbb{C}$.

(a) Describe α and β in terms of radicals involving integers, and deduce that $K = \mathbb{Q}(\alpha, \beta)$.

(b) Show that $[\mathbb{Q}(\beta^2) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\beta) : \mathbb{Q}(\beta^2)] = 2$. Deduce from this that $f(x)$ is irreducible over $\mathbb{Q}$.

(c) Show that $[K : \mathbb{Q}] = 8$ and that $\mathrm{Gal}(K/\mathbb{Q}) \cong D_8$.

8. Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of degree n and let K be the splitting field of $f(x)$ in $\mathbb{C}$. Assume $G = \mathrm{Gal}(K/\mathbb{Q})$ is *abelian*.

(a) Prove that $[K : \mathbb{Q}] = n$ and that $K = \mathbb{Q}(\alpha)$ for every root α of $f(x)$.

(b) Prove that G acts regularly on the set of roots of $f(x)$. (A group acts regularly on a set if it is transitive and the stabilizer of any point is the identity.)

(c) Prove that either all the roots of $f(x)$ are real numbers or none of its roots are real.

9. Let p be a prime and let $n \in \mathbb{Z}^+$ with $(p, n) = 1$. Let K be the splitting field of the polynomial $x^n - 1$ over the finite field $\mathbb{F}_p$ of order p. Prove that $[K : \mathbb{F}_p] = d$, where d is the order of p in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$.