ALGEBRA PH.D. QUALIFYING EXAM

January 10, 2014

A passing paper consists of four problems solved completely plus significant progress on two other problems; moreover, the set of problems solved completely must include one from each of Sections A, B and C.

Section A.

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order 3393 (note that $3393 = 3^2 \cdot 13 \cdot 29$).

(a) Compute the number, n_p, of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3$, 13, and 29.

(b) Show that G contains either a normal Sylow 13-subgroup or a normal Sylow 29-subgroup.

(c) Show that G must have both a normal Sylow 13-subgroup and a normal Sylow 29-subgroup.

(d) Explain briefly why G is solvable.

2. Let G be a finite group and let p be a prime. Assume G has a normal subgroup H of order p.

(a) Prove that H is contained in every Sylow p-subgroup of G.

(b) Prove that if p is the smallest prime dividing the order of G, then H is contained in the center of G.

(c) Prove that if G/H is a simple group, then H is contained in the center of G.

3. Let G be a group and let H be a subgroup of finite index $n > 1$ in G. Let G act by left multiplication on the set of all left cosets of H in G.

(a) Prove that this action is transitive.

(b) Find the stabilizer in G of the identity coset $1H$.

(c) Prove that if G is an infinite group, then it is not a simple group.

Section B.

4. Let R be the following quotient ring of the polynomial ring with rational coefficients:

$$R = \mathbb{Q}[x]/(x^6 - 1).$$

(a) Find all ideals of R. (Be sure to justify that you found them all.)

(b) Determine which of the ideas in (a) are maximal, and for each maximal ideal M describe the quotient ring R/M.

(c) Exhibit an explicit (nonzero) zero divisor in R.

(d) Does R contain any nonzero nilpotent elements? (Briefly justify.)

5. Let R be a Principal Ideal Domain, let M be an R-module, and let p be a nonzero prime in R. Define

$$M_p = \{m \in M \mid p^a m = 0 \text{ for some } a \in \mathbb{Z}^+\} \qquad \text{(called the } p\text{-primary component of } M).$$

- **(a)** Prove that M_p is an R-submodule of M.
- **(b)** Prove that $(M/M_p)_p = 0$, i.e., the p-primary component of M/M_p is zero.
- **(c)** Prove that if q is a nonzero prime in R different from p, then $M_p \cap M_q = 0$.

6. Over the finite field $\mathbb{F}_{17}$ the polynomial $x^{10} - 1$ factors into irreducible polynomials as follows:

$$x^{10} - 1 = (x-1)(x+1)(x^4 + x^3 + x^2 + x + 1)(x^4 - x^3 + x^2 - x + 1).$$

- **(a)** Find, with brief justification, the number of similarity classes of 8×8 matrices A with entries from $\mathbb{F}_{17}$ that satisfy $A^{10} = I$ but $A^i \neq I$ for $1 \leq i \leq 9$.
- **(b)** Exhibit one explicit matrix A satisfying the conditions of (a).
- **(c)** What is the smallest n such that the matrix you found in (b) is similar to a diagonal matrix over the field $\mathbb{F}_{17^n}$?

Section C.

7. Let $\alpha = \sqrt{1 - \sqrt[3]{5}} \in \mathbb{C}$ (where $\sqrt[3]{5}$ denotes the real cube root), let K be the splitting field of the minimal polynomial of α over $\mathbb{Q}$, and let $G = Gal(K/\mathbb{Q})$.

- **(a)** Find the degree of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$.
- **(b)** Show that K contains the splitting field of $x^3 - 5$ over $\mathbb{Q}$ and deduce that G has a normal subgroup H such that $G/H \cong S_3$.
- **(c)** Show that the order of the subgroup H in (b) divides 8.

8. Let K be the splitting field of $x^{61} - 1$ over the finite field $\mathbb{F}_{11}$.

- **(a)** Find the degree of K over $\mathbb{F}_{11}$.
- **(b)** Draw the lattice of all subfields of K *(you need not give generators for these subfields).*
- **(c)** How many elements $\alpha \in K$ generate the multiplicative group $K^\times$?
- **(d)** How many primitive elements are there for the extension $K/\mathbb{F}_{11}$ (i.e., how many β such that $K = \mathbb{F}_{11}(\beta)$)?

9. Let ζ be a primitive 24^{th} root of unity in $\mathbb{C}$, and let $K = \mathbb{Q}(\zeta)$.

- **(a)** Describe the isomorphism type of the Galois group of $K/\mathbb{Q}$.
- **(b)** Determine the number of quadratic extensions of $\mathbb{Q}$ that are subfields of K *(you need not give generators for these subfields).*
- **(c)** Prove that $\sqrt[4]{2}$ is not an element K.