ALGEBRA PH.D. QUALIFYING EXAM

January 13, 2009

A passing paper consists of four problems solved completely plus significant progress on two other problems; moreover, the set of problems solved completely must include one from each of Sections A, B and C.

Section A.

In this section you may quote without proof basic theorems and classifications from group theory, group actions, solvable groups, commutators, etc. as long as you state what facts you are using.

1. Let G be a group of order 10,989 (note that $10989 = 3^3 \cdot 11 \cdot 37$).

- **(a)** Compute the number, n_p, of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 3$, 11, and 37; for each of these n_p give the order of the normalizer of a Sylow p-subgroup.
- **(b)** Show that G contains either a normal Sylow 37-subgroup or a normal Sylow 3-subgroup.
- **(c)** Explain briefly why (in all cases) G has a normal Sylow 11-subgroup.
- **(d)** Deduce that the center of G is nontrivial.

2. Let G be a finite group.

- **(a)** Suppose A and B are normal subgroups of G and both G/A and G/B are solvable. Prove that $G/(A \cap B)$ is solvable.
- **(b)** Deduce from (a) that G has a subgroup that is the unique smallest subgroup with the properties of being normal with solvable quotient — this subgroup is denoted by $G^{(\infty)}$ (i.e., show there is a subgroup $G^{(\infty)} \trianglelefteq G$ with $G/G^{(\infty)}$ is solvable, and if G/N is any solvable quotient, then $G^{(\infty)} \leq N$).

(Remark: For example, when G is solvable, $G^{(\infty)} = 1$; or if G is a perfect group, $G^{(\infty)} = G$.)

- **(c)** If G has a subgroup S isomorphic to A_5 (not necessarily normal), show that $S \leq G^{(\infty)}$.

3. Let G be a group of odd order and let σ be an automorphism of G of order 2.

- **(a)** Prove that for every prime p dividing the order of G there is some Sylow p-subgroup P of G such that $\sigma(P) = P$ (i.e., σ stabilizes the subgroup P — note that σ need not fix P elementwise).
- **(b)** Suppose G is a cyclic group. Prove that $G = A \times B$ where

$$A = C_G(\sigma) = \{g \in G \mid \sigma(g) = g\} \qquad \text{and} \qquad B = \{x \in G \mid \sigma(x) = x^{-1}\}.$$

(Remark: This decomposition is true more generally when G is abelian.)

Section B.

4. Let R be a commutative ring with 1.

- **(a)** Prove that each nilpotent element of R lies in every prime ideal of R.
- **(b)** Assume every nonzero element of R is either a unit or a nilpotent element. Prove that R has a unique prime ideal.

5. Let $R = \mathbb{C}[x, y]$ be the ring of polynomials in the variables x and y, so R may be considered as $\mathbb{C}$-valued functions on (affine) complex 2-space, $\mathbb{C}^2$, in the usual way (R is called the *coordinate ring* of this affine space). Let I be the ideal of all functions in R that vanish on both coordinate axes, i.e., that are zero on the set $\{(a, 0) \mid a \in \mathbb{C}\} \cup \{(0, b) \mid b \in \mathbb{C}\}$. (You may assume I is an ideal.)

(a) Exhibit a set of generators for I. (Be sure to explain briefly why they generate I.)

(b) Show that I is not a prime ideal.

(c) Show that R/I has no nilpotent elements.

6. Classify all finitely generated R-modules, where R is the ring $\mathbb{Q}[x]/(x^2 + 1)^2$.

7. **(a)** Find all possible canonical forms for a matrix over $\mathbb{F}_3$ with characteristic polynomial $x^4 - 1$.

(b) Find all possible canonical forms for a matrix over $\mathbb{F}_2$ with characteristic polynomial $x^4 - 1$.

Section C.

8. Let $K = \mathbb{Q}(\sqrt{3 + \sqrt{5}}\,)$.

(a) Show that $K/\mathbb{Q}$ is a Galois extension.

(b) Determine the Galois group of $K/\mathbb{Q}$.

(c) Find all subfields of K.

9. Let K_1 and K_2 be finite abelian Galois extensions of F contained in a fixed algebraic closure of F. Show that their composite, $K_1 K_2$, is a finite abelian Galois extension of F as well.

10. Let q be a power of a prime, let $Gal(\mathbb{F}_{q^2}/\mathbb{F}_q) = \langle\, \sigma \,\rangle$ (note that σ has order 2). Let N be the usual *norm map* for this extension:

$$N : \mathbb{F}_{q^2}^{\times} \longrightarrow \mathbb{F}_q^{\times} \qquad \text{by} \qquad N(x) = x\,\sigma(x).$$

(a) Prove that N is surjective.

(b) Show that $\mathbb{F}_{q^2}^{\times}$ has an element of order $q + 1$ whose norm is 1.

(c) Find the following index: $|\,\mathbb{F}_q^{\times} : N(\mathbb{F}_q^{\times})\,|$.