Algebra Qualifying Exam — Fall 2019

You have three hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass: Four numbered problems solved completely, with at least one problem from each section, plus substantial progress on two other problems.

MS Pass: Nine lettered subproblems, with at least three in two distinct sections, and at least one letter completed in each section.

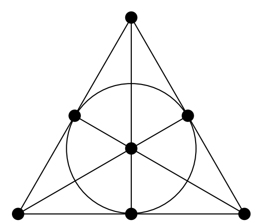
Note: In this exam D_{2n} is the dihedral group of order $2n$ which acts on n elements. If any other notation is confusing, please ask.

Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order 105 and assume that G contains a subgroup N of order 15.
   (a) Explain why N is cyclic.
   (b) Show that if G does not have a normal 7-Sylow subgroup, then N is normal in G.
   (c) Assume N is normal in G. By considering the action of G on N by conjugation, show that N is contained in the center of G, and then show that G is cyclic.

2. Consider the graph depicted below (where the vertices are the solid dots):



   An *automorphism* of a graph is any permutation of vertices that sends edges to edges. Let G be the group of all automorphisms of this graph (the operation is composition).
   (a) Explain why G is isomorphic to a subgroup of S_7, and that G has three orbits in this action.
   (b) Show that the order of G is not divisible by 5 or 7.
   (c) Prove that G is not a simple group.

3. Let $G = D_8 \times S_3$.
   (a) Find the center of G.
   (b) Is G solvable? Explain.
   (c) Exhibit two distinct subgroups of G, both of which are isomorphic to D_8.

Section B

4. Let R be a ring with 1 and let M be a *simple* left R-module (this means that M has no left R-submodules other than 0 and M).

   (a) If $\varphi\colon M \to M$ is a non-trivial R-module homomorphism (i.e. an endomorphism), show that φ is an isomorphism.

   (b) Show that if $m \in M$ with $m \neq 0$, then $M = Rm$.

   (c) Show that there is a left R-module isomorphism $M \cong R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of R.

5. Let $R = \mathbb{R}[x]/(x^4 - 1)$, so R is a commutative ring with 1.

   (a) Show that all ideals of R are principal.

   (b) Find a generator for each maximal ideal of R.

   (c) For each maximal ideal $\mathfrak{m}$, describe an isomorphism from $R/\mathfrak{m}$ to either $\mathbb{R}$ or $\mathbb{C}$.

6. Let R be a commutative ring with 1 which is a subring of the commutative ring S. Let P be a prime ideal of S.

   (a) Show that $P \cap R$ is a prime ideal of R.

   (b) Show that $P[x]$ is a prime ideal of $S[x]$.

   (c) Show that $P[x]$ is not a maximal ideal of $S[x]$.

Section C

7. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and let $\alpha = \sqrt{2} - \sqrt{3}$.

   (a) Show that $[L(\sqrt{\alpha}) : L] = 2$ and $[L(\sqrt{\alpha}) : \mathbb{Q}] = 8$.

   (b) Find the minimal polynomial of $\sqrt{\alpha}$ over $\mathbb{Q}$.

   (c) Show that $L(\sqrt{\alpha})$ is not Galois over $\mathbb{Q}$.

8. Let α be the real, positive fourth root of 5, and let $i = \sqrt{-1} \in \mathbb{C}$. Let $K = \mathbb{Q}(\alpha, i)$.

   (a) Explain why $K/\mathbb{Q}$ is a Galois extension with Galois group dihedral of order 8.

   (b) Find the largest abelian extension of $\mathbb{Q}$ in K (i.e. the unique largest subfield of K that is Galois over $\mathbb{Q}$ with abelian Galois group) — justify your answer.

   (c) Show that $\alpha + i$ is a primitive element for $K/\mathbb{Q}$.

9. Let V be the field of 3^6 elements and let $F \subset V$ be the field of 3 elements, so that V is a 6-dimensional vector space over F. Define

$$T\colon V \to V \quad \text{by} \quad T(a) = a^3 \quad \text{for all } a \in V\ .$$

   (T is called the *Frobenius automorphism of* V.)

   (a) Explain why T is an F-linear transformation from V to itself and $T^6 = I$, where I is the identity linear transformation. (You may quote without proof facts about finite fields and their Galois theory as long as you state these explicitly.)

   (b) Show that $x^6 - 1$ is both the minimal polynomial and the characteristic polynomial for the linear transformation T.
   (Hint: Suppose T satisfies a polynomial of lower degree.)

   (c) Find the Jordan canonical form for the linear transformation T.