Qualifying Exam on Differential Equations January 2020

INSTRUCTIONS: Two problems from each Section must be completed, and one additional problem from each Section must be attempted. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations, but do not need to finish them. Numeric criteria for passing: A problem is considered completed (attempted) if a grade for it is $\geq 85\%$ ($\geq 60\%$).

Time allowed: 3 hours

Section 1

1. Consider the linear inhomogeneous ODE for $y(t)$,

$$y'' + 4y = f(t),$$

where $f(t)$ is a certain given function. Use the method of variation of parameters to obtain the general solution to this equation.

2. For the second-order nonlinear ODE

$$x'' + x' + x - x^3 = 0,$$

(1) convert it into a system of first-order ODEs;
(2) Determine all its fixed points, their type and stability, and then sketch the entire phase portrait.

3. Consider the following two-dimensional system

$$\begin{aligned} x' &= x[x(1-x) - y], \\ y' &= y(x-a), \end{aligned}$$

where a is a parameter. Determine at what a value a Hopf bifurcation occurs.

4. For the weakly nonlinear oscillator problem

$$x'' + x + \epsilon(x^2 - 2)x' = 0, \quad x(0) = 1, x'(0) = 0,$$

where $\epsilon \ll 1$,
(1) Use multi-scale perturbation method to determine its leading-order solution;
(2) As $t \to \infty$, where does this leading-order solution approach?

Some helpful integrals: defining $\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$, then

$$\langle \cos^{2n+1}\theta \rangle = \langle \sin^{2n+1}\theta \rangle = 0, \quad \langle \sin^n\theta \cos\theta \rangle = \langle \sin\theta \cos^n\theta \rangle = 0,$$

$$\langle \cos^2\theta \rangle = \langle \sin^2\theta \rangle = \frac{1}{2}, \quad \langle \sin^2\theta \cos^2\theta \rangle = \frac{1}{8}, \quad \langle \sin^{2n}\theta \rangle = \langle \cos^{2n}\theta \rangle = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}.$$

Section 2

5. (a) Let $f(s)$ and $g(s)$ be square integrable on the infinite line, so that their Fourier transforms $F[f](\omega)$ and $F[g](\omega)$ exist, where

$$F[f](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega s} f(s) ds\,,$$

and similarly for $F[g]$. **Show that**

$$F[f * g] = F[f] \cdot F[g]\,,$$

where

$$f * g\,(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s - s_1) g(s_1) ds_1\,.$$

(b) Consider the Heat equation on the infinite line:

$$u_t = u_{xx}, \qquad u(x, 0) = q(x)\,,$$

and $q \to 0$ sufficiently fast as $|x| \to \infty$. **Use the result of part (a) to show that**

$$u(x, t) = \int_{-\infty}^{\infty} \phi(x - x_1, t) q(x_1) dx_1\,,$$

where

$$\phi(x, t) = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/(4t)}\,.$$

Note: You will need the value of the following integral:

$$\int_{-\infty}^{\infty} e^{-\omega^2 a + i\omega b} d\omega = \sqrt{\frac{\pi}{a}}\, e^{-b^2/(4a)}\,, \qquad a > 0\,.$$

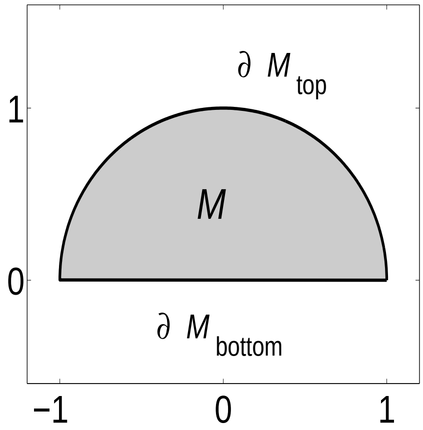
6. Use the Method of Characteristics to obtain the solution at $t = 12$ of the following initial-value problem:

$$w_t + \frac{1}{2 + 2x}\, w_x = 0, \qquad -\infty < x < \infty, \quad t > 0;$$

$$w(x, 0) = \begin{cases} 1, & 1 \le x \le 2; \\ 0, & \text{elsewhere.} \end{cases}$$

Note: The coefficients are designed so that the most complicated equation that you will need to solve is quadratic.

7. Find the temperature $u(r, \theta, t)$ of a semi-circular membrane (see the figure below) which satisfies the following initial-boundary-value problem:



$$u_t = u_{rr} + \frac{1}{r}\,u_r + \frac{1}{r^2}\,u_{\theta\theta}, \qquad r,\theta \ \text{ inside } \ M \tag{1}$$

$$u_\theta = 0, \qquad r,\theta \ \text{ on } \ \partial M_{\text{bottom}} \tag{2}$$

$$u = 0, \qquad r,\theta \ \text{ on } \ \partial M_{\text{top}} \tag{3}$$

$$u(r,\theta,0) = f(r,\theta), \qquad r,\theta \ \text{ inside } \ M\,. \tag{4}$$

8. Solve the initial-boundary-value problem

$$u_{tt} = u_{xx} + \sin(\pi x), \qquad 0 < x < 1, \quad t > 0;$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = u_t(x,0) = 0\,.$$

To receive full credit, you *must explicitly* indicate all steps in your solution where you require the orthogonality relation among the eigenfunctions.