Section 1, ODE

1. Draw the phase portrait for the system

\[
\begin{align*}
\dot{x} &= x(2 - x - y) \\
\dot{y} &= x - y
\end{align*}
\]

and identify the fixed points and their stability.

2. Solve the non-homogeneous linear system

\[
\begin{bmatrix}
1 & 1 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} = \begin{bmatrix}
t \\
1
\end{bmatrix}
\]

with the initial condition \(\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T\).

3. Express the linear system of ODEs

\[
\begin{align*}
\dot{x}_1 &= ax_1 - bx_2 \\
\dot{x}_2 &= bx_1 + ax_2
\end{align*}
\]

in polar coordinates, where \(r^2 = x_1^2 + x_2^2\) and \(\theta = \tan^{-1}(x_2/x_1)\). The result should have a very simple form. Then solve using the initial conditions \(r(0) = r_0, \theta(0) = \theta_0\).

4. Consider the biased van der Pol oscillator \(\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a\). Find the curves in \((\mu, a)\) space at which Hopf bifurcations occur.
Section 2. PDE

5. For $0 \leq x \leq \pi$, solve the problem

\[
\phi_t = \phi_{xx} + w(x, t),
\]

\[
\phi(0, t) = 0, \quad \phi_x(\pi, t) = 0,
\]

\[
\phi(x, 0) = f(x).
\]

6. Solve the following 2D heat equation on a circular disk as simply as possible:

\[
u_t = \nabla^2 u,
\]

\[
u(a, \theta, t) = 0,
\]

\[
u(r, \theta, 0) = f(r).
\]

Here $a$ is the radius of the disk, and $f(r)$ is a prescribed arbitrary function.

7. Use the method of characteristics to solve the problem

\[
\rho_t - x \rho_x = \rho + t, \quad -\infty < x < \infty,
\]

\[
\rho(x, 0) = f(x),
\]

and express your solution explicitly in terms of the function $f(x)$.

8. Consider the following eigenvalue problem,

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{d \phi}{dr} \right) + \left( \lambda - \frac{1}{r^2} \right) \phi = 0, \quad 0 < r \leq 3,
\]

\[
\phi(0) \text{ is finite}; \quad \phi(3) = 0.
\]

(a) Rewrite this eigenvalue problem in the Sturm-Liouville form;

(b) Prove that its eigenfunctions of different eigenvalues are orthogonal to each other under a certain weight. What is this weight?

(c) Determine these eigenvalues and eigenfunctions.