COMBINATORICS QUALIFYING EXAM
January 2022

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions, each with three parts (a,b,c).

PhD Pass: Three numbered questions solved completely, or two solved completely with substantial progress on another two.

MS Pass: Substantial progress on three questions.

Question 1

(a) State Menger's Theorem (in the context of graph vertex-connectivity).

(b) Let G be a simple k-vertex-connected graph and v a vertex of G. Let C be a cycle in G of length $l \leq k$, that does not contain v. Show that there is a set of l vw-paths, one for each vertex w on C, that are disjoint except at v.

(c) Show that if G is k-vertex-connected, it must contain a cycle of length $> k$.

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Question 2

(a) Give the definition of the the Turán number or extremal number $\text{ex}(n, H)$, where n is a positive integer and H is a graph.

(b) Determine the value of $\text{ex}(n, P_4)$ for all n, where P_4 is the path graph on 4 vertices (and 3 edges). If it is convenient, you may assume that $n \equiv 0 \pmod{k}$ for an appropriate natural number k.

(c) Give an example of a graph that is edge-maximal without a P_4 subgraph, but not extremal.

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Question 3

Let c_n be the (maximum) number of regions on a piece of paper determined by n intersecting circles. *Note: Each circle must intersect each other circle; no two circles are tangent; assume no three circles intersect at a point; include "infinite" regions.*

(a) Gather data by drawing pictures for $n \leq 4$ circles and computing c_n directly.

(b) Find a recurrence relation for c_n (include initial conditions!).

(c) Use generating functions to find and prove a closed formula for c_n, $n \geq 0$.

Aside (not for credit): What does your answer tell you about Venn diagrams?

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Question 4

(a) State the Pieri rule for h_n.

(b) Use the Pieri rule to find the Schur expansion of the $h_{(2,1,3)}$.

(c) How do the Kostka numbers arise in your answer to part (b)?

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Question 5

(a) Give the independent set axioms for a matroid.

(b) Give the circuit axioms for a matroid.

(c) Given a ground set E and a collection $\mathcal{I}$ of subsets of E which satisfy the independent sets axioms, define the circuits of the matroid in terms of $\mathcal{I}$, and prove that they satisfy the circuit axioms.

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Question 6

(a) Give the vertex definition of a convex polytope.

(b) Give the half-space definition of a convex polytope.

(c) Let M be a uniform matroid $\mathcal{U}_n^r$, and let P_M be the associated matroid polytope. Give both vertex and half-space presentations of P_M. Remark: These polytopes are also known as *hypersimplices*.

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