You have three hours to complete this exam. When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

**PhD Pass:** Four numbered questions solved completely, with at least one from each section.

**MS Pass:** Substantial progress on three questions, in any section.
Section A

Question 1

Let $G$ be a forest.

(a) Prove that $G$ has $n-c$ edges, where $n$ is the number of vertices and $c$ the number of connected components of $G$.

(b) Find the average degree of $G$.

(c) Let $G_1, G_2, \ldots, G_k$ be connected subgraphs of $G$. Prove that their intersection is either empty or a tree.

Question 2

(a) State and prove Hall’s Theorem.

(b) Let $G$ be bipartite graph with bipartition $V \cup W$, and maximum degree $\Delta(G) \geq 1$. Let $S_V$ be the set of all vertices $v \in V$ such that $d(v) = \Delta(G)$, and let $S_W$ be defined similarly. Show that $G$ has a matching that saturates $S_V$ and a matching that saturates $S_W$.

(c) Show that $G$ has a matching that saturates $S_V \cup S_W$.

Question 3

(a) State Turán’s Theorem.

(b) Give an example of a graph $G$ that is edge-maximal without a $K_3$ subgraph, but not extremal.

(c) Determine the value of $\text{ex}(n, K_{1,3})$ for all $n \in \mathbb{N}$.

Section B

Question 1

Let $f_n$ denote the $n$-th Fibonacci number (using the standard convention that $f_0 = 0$ and $f_1 = 1$).

(a) Write down a functional equation for the ordinary generating function for $\{f_n\}_{n \geq 0}$.

(b) Write down a functional equation for the exponential generating function for $\{f_n\}_{n \geq 0}$.

(c) Solve either functional equation.
**Question 2**

(a) State the Robinson-Schensted correspondence.

(b) Apply the correspondence to the permutation $\sigma = [4, 6, 1, 2, 5, 3]$ (written in one-line notation).

(c) Explain how the images of $\sigma$ and $\sigma^{-1}$ are related; illustrate by a direct computation applied to $\sigma^{-1}$.

**Section B**

**Question 1**

Let $M$ be the matroid with ground set $\{a, b, c, d, e, f\}$ and circuits

$$\{\{a, b, e\}, \{b, c, d\}, \{d, f, e\}, \{a, c, f\}, \{a, e, d, c\}, \{a, b, d, f\}, \{b, c, f, e\}\}.$$  

(a) Describe a graphic representation of $M$, and prove that the matroid is self-dual. Explain why this matroid is irreducible. How many independent sets does this matroid have?

(b) Draw the lattice of flats of $M$, and apply the Möbius function to calculate the characteristic polynomial $\chi_M(q)$. Evaluate this polynomial at 3, and explain the combinatorial meaning of this value in terms of the graph.

**Question 2**

Let $P$ be a polygon.

(a) Prove that every triangulation of $P$ is regular (coherent). How many vertices does the secondary polytope of $P$ have?

(b) Characterize when $P$ is a zonotope. Now, suppose $P$ is a zonotope. How many parallelograms will a fine tiling (paving) of $P$ have? Explain why the simple matroid associated to an octagonal zonotope cannot be graphic.