You have three hours to complete this exam. When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

**PhD Pass:** Four numbered problems solved completely, with at least one problem from each section, plus substantial progress on two other problems.

**MS Pass:** Nine lettered subproblems, with at least three in two distinct sections, and at least one letter completed in each section.

Note: In this exam \( D_{2n} \) is the dihedral group of order \( 2n \) which acts on \( n \) elements. If any other notation is confusing, please ask.
Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let $G$ be a group of order 105 and assume that $G$ contains a subgroup $N$ of order 15.
   (a) Explain why $N$ is cyclic.
   (b) Show that if $G$ does not have a normal 7-Sylow subgroup, then $N$ is normal in $G$.
   (c) Assume $N$ is normal in $G$. By considering the action of $G$ on $N$ by conjugation, show that $N$ is contained in the center of $G$, and then show that $G$ is cyclic.

2. Consider the graph depicted below (where the vertices are the solid dots):

   An automorphism of a graph is any permutation of vertices that sends edges to edges. Let $G$ be the group of all automorphisms of this graph (the operation is composition).
   (a) Explain why $G$ is isomorphic to a subgroup of $S_7$, and that $G$ has three orbits in this action.
   (b) Show that the order of $G$ is not divisible by 5 or 7.
   (c) Prove that $G$ is not a simple group.

3. Let $G = D_8 \times S_3$.
   (a) Find the center of $G$.
   (b) Is $G$ solvable? Explain.
   (c) Exhibit two distinct subgroups of $G$, both of which are isomorphic to $D_8$. 
Section B

4. Let $R$ be a ring with 1 and let $M$ be a simple left $R$-module (this means that $M$ has no left $R$-submodules other than 0 and $M$).

(a) If $\varphi : M \to M$ is a non-trivial $R$-module homomorphism (i.e. an endomorphism), show that $\varphi$ is an isomorphism.

(b) Show that if $m \in M$ with $m \neq 0$, then $M = Rm$.

(c) Show that there is a left $R$-module isomorphism $M \cong R/m$ for some maximal left ideal $m$ of $R$.

5. Let $R = \mathbb{R}[x]/(x^4 - 1)$, so $R$ is a commutative ring with 1.

(a) Show that all ideals of $R$ are principal.

(b) Find a generator for each maximal ideal of $R$.

(c) For each maximal ideal $m$, describe an isomorphism from $R/m$ to either $\mathbb{R}$ or $\mathbb{C}$.

6. Let $R$ be a commutative ring with 1 which is a subring of the commutative ring $S$. Let $P$ be a prime ideal of $S$.

(a) Show that $P \cap R$ is a prime ideal of $R$.

(b) Show that $P[x]$ is a prime ideal of $S[x]$.

(c) Show that $P[x]$ is not a maximal ideal of $S[x]$.
Section C

7. Let \( L = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) and let \( \alpha = \sqrt{2} - \sqrt{3} \).

(a) Show that \([L(\sqrt{\alpha}) : L] = 2\) and \([L(\sqrt{\alpha}) : \mathbb{Q}] = 8\).

(b) Find the minimal polynomial of \( \sqrt{\alpha} \) over \( \mathbb{Q} \).

(c) Show that \( L(\sqrt{\alpha}) \) is not Galois over \( \mathbb{Q} \).

8. Let \( \alpha \) be the real, positive fourth root of 5, and let \( i = \sqrt{-1} \in \mathbb{C} \). Let \( K = \mathbb{Q}(\alpha, i) \).

(a) Explain why \( K/\mathbb{Q} \) is a Galois extension with Galois group dihedral of order 8.

(b) Find the largest abelian extension of \( \mathbb{Q} \) in \( K \) (i.e. the unique largest subfield of \( K \) that is Galois over \( \mathbb{Q} \) with abelian Galois group) — justify your answer.

(c) Show that \( \alpha + i \) is a primitive element for \( K/\mathbb{Q} \).

9. Let \( V \) be the field of \( 3^6 \) elements and let \( F \subset V \) be the field of 3 elements, so that \( V \) is a 6-dimensional vector space over \( F \). Define

\[ T: V \to V \text{ by } T(a) = a^3 \text{ for all } a \in V. \]

\( T \) is called the Frobenius automorphism of \( V \).

(a) Explain why \( T \) is an \( F \)-linear transformation from \( V \) to itself and \( T^6 = I \), where \( I \) is the identity linear transformation. (You may quote without proof facts about finite fields and their Galois theory as long as you state these explicitly.)

(b) Show that \( x^6 - 1 \) is both the minimal polynomial and the characteristic polynomial for the linear transformation \( T \).

(Hint: Suppose \( T \) satisfies a polynomial of lower degree.)

(c) Find the Jordan canonical form for the linear transformation \( T \).