There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let \( \mathbb{D} \) denote the open disc of radius 1 centered at the origin.

1. Let \( f \) be holomorphic on a connected open set \( U \). Prove that if \( f(z)^2 = \overline{f(z)} \) for all \( z \in U \) then \( f \) is constant on \( U \). Find all possible values for \( f \).

2. Let \( \gamma \) be the circle of radius 5 centered at 0. Evaluate with brief justification the integrals:
   \[
   (a) \int_{\gamma} \frac{z}{z-1} \, dz, \quad (b) \int_{\gamma} e^{1/z} \, dz.
   \]

3. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point \( z = 2 \) for the following function (explain briefly how the inner and outer radii of the annulus are determined):
   \[
f(z) = \frac{z}{1-z^2} + \frac{6}{(z-4)^2}.
   \]

4. Use the calculus of residues to evaluate the improper integral
   \[
   \int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1} \, dx.
   \]

5. Prove that if \( f \) is entire and there are positive real numbers \( A, B \) and \( k \) such that \( |f(z)| \leq A + B|z|^{k} \) for all \( z \in \mathbb{C} \), then \( f \) is a polynomial.

6. Let \( f \) be analytic on the closed unit disc \( \overline{\mathbb{D}} \), and assume \( |f(z)| < 1 \) on its boundary. Prove that there is one and only one point \( z_0 \in \mathbb{D} \) such that \( f(z_0) = z_0 \).

7. (a) Exhibit an entire function, \( P(z) \), that has simple zeros at the numbers \( \sqrt{n} \) for each positive integer \( n \), and no other zeros.

    (b) For the function \( P \) you gave in part (a), describe \( P'/P \) as an infinite series (not necessarily a Taylor series however).
8. Define \( f(z) = \int_0^1 \frac{dt}{1 + tz} \).

(a) Show by using Morera’s Theorem that \( f \) is analytic on the open unit disc \( \mathbb{D} \).

(b) Find a power series expansion for \( f(z) \) valid on \( \mathbb{D} \).

9. Let \( P(z) \) and \( Q(z) \) be polynomials with degree \( Q \geq \text{degree } P + 2 \). Prove that

\[
\sum_{z_i} \text{Res}_{z=z_i} \frac{P(z)}{Q(z)} = 0
\]

where the sum is over all poles \( z_i \) in \( \mathbb{C} \) of the rational function \( \frac{P}{Q} \).

10. Suppose \( f \) is analytic on the punctured unit disc \( \mathbb{D} - \{0\} \) and the real part of \( f \) is positive there. Prove that \( f \) has a removable singularity at 0.