There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others.

1. Find all solutions (if any) to the equation $i^z = 2$, where $i = \sqrt{-1}$.

2. Let $C_1$ and $C_2$ be the circles centered at the origin in $\mathbb{C}$, of radii 1 and 2 respectively. Evaluate, with brief justifications, the integrals,
   
   (a) $\int_{C_1} \frac{e^z}{z - 1 - i} \, dz$
   
   (b) $\int_{C_2} \frac{e^z}{z - 1 - i} \, dz$.

3. Use the calculus of residues to evaluate the improper integral
   
   $\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} \, dx$.

4. Find each singularity in $\mathbb{C}$ and classify it (removable, pole of order $n$, essential) for the function
   
   $f(z) = \frac{z}{(4 + z^2) \sin(\pi z)}$.

5. Find a Laurent series expansion valid in the annulus $\{ z : 1 < |z| < 2 \}$ for the following function:
   
   $f(z) = \frac{z^2}{(z + 1)(z - 2)}$.

6. Suppose $f : \mathbb{C} \to \mathbb{C}$ has the property that for each $z \in \mathbb{C}$ there is an open disc $D_z$ centered at $z$ such that $f$ is a polynomial on $D_z$ (where the radius of $D_z$ and the polynomial $f|_{D_z}$ both may depend on $z$). Prove that $f$ is a polynomial on all of $\mathbb{C}$ (i.e., $f$ is the same polynomial on each $D_z$).

7. Show that $z^5 + 3z^3 + 7$ has exactly five zeros in the disc $|z| < 2$.

8. Suppose $D$ is the unit disc centered at the origin in $\mathbb{C}$ and, for each natural number $n$, $f_n : D \to \mathbb{C}$ is an analytic function. Prove that if the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$ on $D$ then $f$ is analytic.

9. Suppose $f$ and $g$ are entire functions such that $|f(z)| \leq |g(z)|$ at all points $z$ where $g(z) \neq 0$. Prove that $f = cg$ for some constant $c \in \mathbb{C}$.

10. Exhibit an analytic, one-to-one, onto map from $\{ z : |z| < 1, \ \text{Re}(z) > 0 \}$ to $\{ z : |z| < 1 \}$. You may express your map as a sequence of compositions.