COMPLEX VARIABLES PH.D. QUALIFYING EXAM

September 28, 2007

There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others.

1. Find all solutions (if any) to the equation $i^z = 2$, where $i = \sqrt{-1}$.

2. Let C_1 and C_2 be the circles centered at the origin in $\mathbb{C}$, of radii 1 and 2 respectively. Evaluate, with brief justifications, the integrals,

   (a) $\displaystyle\int_{C_1} \frac{e^z}{z-1-i}\,dz$ (b) $\displaystyle\int_{C_2} \frac{e^z}{z-1-i}\,dz.$

3. Use the calculus of residues to evaluate the improper integral
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx.$$

4. Find each singularity in $\mathbb{C}$ and classify it (removable, pole of order n, essential) for the function
$$f(z) = \frac{z}{(4+z^2)\sin(\pi z)}.$$

5. Find a Laurent series expansion valid in the annulus $\{z : \ 1 < |z| < 2\}$ for the following function:
$$f(z) = \frac{z^2}{(z+1)(z-2)}.$$

6. Suppose $f : \mathbb{C} \to \mathbb{C}$ has the property that for each $z \in \mathbb{C}$ there is an open disc D_z centered at z such that f is a polynomial on D_z (where the radius of D_z and the polynomial $f|_{D_z}$ both may depend on z). Prove that f is a polynomial on all of $\mathbb{C}$ (i.e., f is the same polynomial on each D_z).

7. Show that $z^5 + 3z^3 + 7$ has exactly five zeros in the disc $|z| < 2$.

8. Suppose D is the unit disc centered at the origin in $\mathbb{C}$ and, for each natural number n, $f_n : D \to \mathbb{C}$ is an analytic function. Prove that if the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on D then f is analytic.

9. Suppose f and g are entire functions such that $|f(z)| \le |g(z)|$ at all points z where $g(z) \ne 0$. Prove that $f = cg$ for some constant $c \in \mathbb{C}$.

10. Exhibit an analytic, one-to-one, onto map from $\{z : |z| < 1,\ \mathrm{Re}(z) > 0\}$ to $\{z : |z| < 1\}$. You may express your map as a sequence of compositions.