A passing paper consists of a total of six problems done completely correctly, or five problems done correctly with substantial progress on two others. At least three problems from each of Section A (Real Analysis) and Section B (Complex Analysis) must count toward the passing criteria, and two of these from each section must be completely correct.

Section A. Real Analysis

1. Let \((M, d)\) be a metric space, and let \(\{x_n\}_{n=1}^{\infty}\) and \(\{y_n\}_{n=1}^{\infty}\) be Cauchy sequences in \(M\). 
Prove that the sequence of real numbers \(\{d(x_n, y_n)\}_{n=1}^{\infty}\) converges in \(\mathbb{R}\). 
(Do not assume \(M\) is complete.)

2. Let \(f_n : \mathbb{R} \to \mathbb{R}\) be a sequence of functions that converges uniformly to a function \(f : \mathbb{R} \to \mathbb{R}\). 
Prove that if the sequence of real numbers \(\{a_n\}_{n=1}^{\infty}\) converges to \(a\) and \(f\) is continuous at \(a\), then the sequence \(\{f_n(a_n)\}_{n=1}^{\infty}\) converges to \(f(a)\).

3. For each real number \(t > 0\) let \(F(t) = \int_0^{\infty} \frac{e^{-xt}}{1 + x^2} \, dx\). (You may treat the integrals as either Riemann or Lebesgue — whichever you prefer.)
   (a) Show that \(F(t)\) is defined (i.e., converges) for every \(t > 0\).
   (b) Prove that \(F\) is continuous on \((0, \infty)\).

4. Let \(\overline{\mu}\) be an outer measure on a set \(X\). Show that a subset \(E\) of \(X\) is \(\overline{\mu}\)-measurable if and only if for every natural number \(n\) there is a measurable set \(E_n\) with \(E_n \subseteq E\) and \(\overline{\mu}(E - E_n) < \frac{1}{n}\).

5. Let \((X, \mathcal{M}, \mu)\) be a measure space. We say that \(\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}\) almost fills up \(X\) if, for all \(A \in \mathcal{M}\) with finite measure, 
\[ \lim_{n \to \infty} \mu(A \setminus E_n) = 0. \]
Show that \(\{E_n\}_{1}^{\infty} \subseteq \mathcal{M}\) almost fills up \(X\) if and only if for all \(f \in L^1(X, \mathcal{M}, \mu), f\chi_{E_n} \to f\) in \(L^1(X)\).

6. Find, with justification, the value of 
\[ \lim_{n \to \infty} \int_1^{\infty} \frac{n \sin(x^2/n)}{x^4} \, dx. \]

7. Let \(F : \mathbb{R}^4 \to \mathbb{R}^2\) by \(F(x, y, u, v) = (x^3 + vx + y, uy + v^3 - x)\).
   (a) Find the Jacobian matrix of \(F\) at an arbitrary point in the domain.
   (b) At what points satisfying \(F(x, y, u, v) = (0, 0)\) does the Implicit Function Theorem allow you to solve for \(u\) and \(v\) in terms of \(x\) and \(y\) ?
   (c) At any one of the points in part (a) of your choosing compute \(\partial u / \partial x\).
Section B. Complex Analysis

8. Identify explicitly the real and imaginary parts of the function \( f(z) = z \cos z \), and verify any one of the Cauchy–Riemann equations for \( f \) at an arbitrary point \( z \).

9. Use the method of residues to find the value of the integral \( \int_0^\infty \frac{x^2}{x^6 + 1} \, dx \).

10. Find the Laurent series of the form \( \sum_{n=-\infty}^{\infty} c_n z^n \) for \( f(z) = \frac{33}{(2z - 1)(z + 5)} \) that converges in an annulus containing the point \( z = -3i \), and state precisely where this Laurent series converges.

11. Use Rouché’s Theorem to determine the number of zeros of \( f(z) = 2z^5 - 6z^2 + z + 1 \) in the annulus \( 1 \leq |z| \leq 2 \).

12. Use any method to find the value of \( \int_C \tan z \, dz \), where \( C \) is the circle of radius 8 centered at the origin, oriented counterclockwise.

13. Describe explicitly all entire functions \( f(z) \) that satisfy the following inequality:

\[ |f(z)| \leq |e^z \sin z|, \quad \text{for all } z \in \mathbb{C}. \]

14. Let \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \) be the unit disk in the complex plane, and let \( f_n : D \to D \) be a sequence of analytic functions that converges pointwise to \( f : D \to \mathbb{C} \). Prove that \( f \) is analytic. (You may quote results from both real and complex analysis.)