REAL AND COMPLEX VARIABLES PH.D. QUALIFYING EXAM

September 17, 2015

Three Hours

A passing paper consists of a total of six problems done completely correctly, or five problems done correctly with substantial progress on two others. At least three problems from each of Section A (Real Analysis) and Section B (Complex Analysis) must count toward the passing criteria, and two of these from each section must be completely correct.

Section A. Real Analysis

1. Let (M,d) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be Cauchy sequences in M. Prove that the sequence of real numbers $\{d(x_n,y_n)\}_{n=1}^{\infty}$ converges in $\mathbb{R}$.
(Do not assume M is complete.)

2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions that converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Prove that if the sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ converges to a and f is continuous at a, then the sequence $\{f_n(a_n)\}_{n=1}^{\infty}$ converges to $f(a)$.

3. For each real number $t > 0$ let $F(t) = \displaystyle\int_0^{\infty} \frac{e^{-xt}}{1+x^2}\,dx$. (You may treat the integrals as either Riemann or Lebesgue — whichever you prefer.)
   (a) Show that $F(t)$ is defined (i.e., converges) for every $t > 0$.
   (b) Prove that F is continuous on $(0,\infty)$.

4. Let $\overline{\mu}$ be an outer measure on a set X. Show that a subset E of X is $\overline{\mu}$-measurable if and only if for every natural number n there is a measurable set E_n with $E_n \subseteq E$ and $\overline{\mu}(E - E_n) < \frac{1}{n}$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ *almost fills up* X if, for all $A \in \mathcal{M}$ with finite measure,
$$\lim_{n\to\infty} \mu(A \setminus E_n) = 0.$$
Show that $\{E_n\}_1^{\infty} \subseteq \mathcal{M}$ almost fills up X if and only if for all $f \in L^1(X, \mathcal{M}, \mu)$, $f\chi_{E_n} \to f$ in $L^1(X)$.

6. Find, with justification, the value of
$$\lim_{n\to\infty} \int_1^{\infty} \frac{n\sin(x^2/n)}{x^4}\,dx.$$

7. Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ by $F(x,y,u,v) = (x^3 + vx + y\,,\; uy + v^3 - x)$.
   (a) Find the Jacobian matrix of F at an arbitrary point in the domain.
   (b) At what points satisfying $F(x,y,u,v) = (0,0)$ does the Implicit Function Theorem allow you to solve for u and v in terms of x and y ?
   (c) At any *one* of the points in part (a) of your choosing compute $\partial u/\partial x$.

Section B. Complex Analysis

8. Identify explicitly the real and imaginary parts of the function $f(z) = z\cos z$, and verify any *one* of the Cauchy–Riemann equations for f at an arbitrary point z.

9. Use the method of residues to find the value of the integral $\displaystyle\int_0^\infty \frac{x^2}{x^6+1}\,dx$.

10. Find the Laurent series of the form $\displaystyle\sum_{n=-\infty}^{\infty} c_n z^n$ for $f(z) = \dfrac{33}{(2z-1)(z+5)}$ that converges in an annulus containing the point $z = -3i$, and state precisely where this Laurent series converges.

11. Use Rouché's Theorem to determine the number of zeros of $f(z) = 2z^5 - 6z^2 + z + 1$ in the annulus $1 \le |\,z\,| \le 2$.

12. Use any method to find the value of $\displaystyle\int_C \tan z\,dz$, where C is the circle of radius 8 centered at the origin, oriented counterclockwise.

13. Describe explicitly all entire functions $f(z)$ that satisfy the following inequality:

$$|\,f(z)\,| \le |\,e^z \sin z\,|, \qquad \text{for all } z \in \mathbb{C}.$$

14. Let $D = \{z \in \mathbb{C} \mid |\,z\,| < 1\}$ be the unit disk in the complex plane, and let $f_n : D \to D$ be a sequence of analytic functions that converges pointwise to $f : D \to \mathbb{C}$. Prove that f is analytic. (You may quote results from both real and complex analysis.)