REAL AND COMPLEX VARIABLES PH.D. QUALIFYING EXAM

May 10, 2018

Three Hours

A passing paper consists of a total of six problems done completely correctly, or five problems done correctly with substantial progress on two others. At least three problems from each of Section A (Real Analysis) and Section B (Complex Analysis) must count toward the passing criteria, and two of these from each section must be completely correct.

Section A. Real Analysis

1. Let X be any topological space and let $X \times X$ be the space with the product topology. Let $D = \{(x, x) \mid x \in X\}$ be a diagonal in $X \times X$. Prove any *one* implication of the statement: X is a Hausdorff space if and only if D is closed in $X \times X$.

2. Assume $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Prove that there exist some real numbers y_0 and y_1 such that if we extend f at the two end points by defining

$$f(0) = y_0 \qquad \text{and} \qquad f(1) = y_1$$

then this new function is continuous (hence uniformly continuous) on the closed interval [0,1]. (Note that y_0 and y_1 are then necessarily unique.)

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \dfrac{a}{b}, \text{ a rational number in lowest terms } \textit{with } b \textit{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Determine whether or not f is Riemann integrable on [0,1]; and if so, find its Riemann integral. (You may use either Riemann's Condition or Lebesgue's Theorem.)

   (b) Explain briefly why f is Lebesgue integrable, and find its Lebesgue integral $\int_0^1 f$.

4. Let (X, Λ, μ) be a measure space, where μ is a measure on the σ-algebra Λ. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on X. Prove that the set

$$E = \{x \in X \mid \lim_{n \to \infty} f_n(x) = +\infty\}$$

is a measurable subset of X.

5. Find, with justification, the value of

$$\lim_{N \to \infty} \int_0^1 \left(\sum_{n=0}^{N} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{N} \frac{(-2x)^n}{n!} \right) dx.$$

6. Let $f : [0, 1] \to [0, \infty)$ be continuous.

   (a) Prove that $\displaystyle\lim_{n \to \infty} \int_0^1 x^n f(x)\, dx = 0.$

   (b) Prove that $\displaystyle\lim_{n \to \infty} \int_0^1 (n + 1) x^n f(x)\, dx = f(1).$

Section B. Complex Analysis

7. Exhibit an analytic isomorphism from the horizontal strip $\{z \mid -1 < \text{Im } z < 1\}$ to the open unit disc $\{z \mid |z| < 1\}$. (Briefly show any steps if you use compositions of "well known" conformal maps.)

8. Use the method of residues to find the value of the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2(x^2+9)}\,dx$. (Sketch your contour of integration and very briefly justify your method.)

9. Find the Laurent series of the form $\sum_{n=-\infty}^{\infty} c_n z^n$ for $f(z) = \dfrac{6}{z^2+9} - \dfrac{9}{(z+6)^2}$ that converges in an annulus containing the point $z = -4i$, and state precisely where this Laurent series converges.

10. Let $\mathbb{D}^{**}$ be the open unit disc $D(0,1)$ with the two points 0 and $\frac{1}{2}$ removed. Find all analytic automorphisms of $\mathbb{D}^{**}$ (i.e., bijections $f : \mathbb{D}^{**} \to \mathbb{D}^{**}$ that are analytic with analytic inverse). (You may quote standard theorems about analytic automorphisms.)

11. Let $p(z)$ and $q(z)$ be non-constant polynomials over $\mathbb{C}$ of different degrees. Prove that there is no entire function $f(z)$ such that

$$|\,p(z)\,| \le |\,f(z)\,| \le |\,q(z)\,|, \qquad \text{for all } z \in \mathbb{C}.$$

12. Prove that the series

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{(z-n)^3}$$

converges to a meromorphic function on all of $\mathbb{C}$, and find its poles and each of their orders.