A passing paper consists of a total of six problems done completely correctly, or five problems done correctly with substantial progress on two others. At least three problems from each of Section A (Real Analysis) and Section B (Complex Analysis) must count toward the passing criteria, and two of these from each section must be completely correct.

Section A. Real Analysis

1. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be defined by
\[
 f(x) = \begin{cases} 
 \frac{a}{b^2} & \text{if } x = \frac{a}{b}, \text{ a rational number in lowest terms}, \\
 0 & \text{otherwise}. 
\end{cases}
\]
Explain why \( f \) is Riemann integrable, and find \( \int_0^1 f \).

2. Find with justification the limit: \( \lim_{n \to \infty} \int_1^\infty e^{-ntn} dt \).

3. Let \((X, \Lambda, \mu)\) be a measure space, where \( \mu \) is a measure on the \( \sigma \)-algebra \( \Lambda \). A family \( F \) of real valued, measurable functions on \( X \) is called uniformly integrable if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that
\[
\text{for every } E \in \Lambda \text{ with } \mu(E) < \delta \text{ we have } \int_X |f|\chi_E \, d\mu < \epsilon, \quad \text{for all } f \in F
\]
where \( \chi_E \) is the characteristic function of \( E \) (i.e., this \( \delta \) works uniformly for all \( f \in F \)).
For any \( p \) with \( 1 < p \leq \infty \), prove that the family of all measurable, real valued functions \( f \) on \( X \) that satisfy \( ||f||_p \leq 1 \) is a uniformly integrable family (i.e., the closed unit ball centered at \( 0 \) in \( L^p(X) \) is uniformly integrable).

4. Let \((X, \Lambda, \mu)\) be a measure space, where \( \mu \) is a measure on the \( \sigma \)-algebra \( \Lambda \). Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of measurable functions on \( X \). Prove that the set
\[
\{x \in X \mid \{f_n(x)\}_{n=1}^{\infty} \text{ is a bounded sequence}\}
\]
is a measurable subset of \( X \).

5. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by
\[
f(x, y) = (x^3 - 3xy^2, 3x^2y - y^3).
\]
(a) Find the Jacobian matrix of \( f \) at an arbitrary point of \( \mathbb{R}^2 \).
(b) Find all points \((a, b)\) in the domain of \( f \) where \( f \) has a local inverse. (Justify your answer.)
6. Let $X$ be a subset of the real line and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions on $X$.

(a) Give the definition of what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ to converge uniformly on $X$.

(b) Give the definition of what it means for the series $\sum_{n=1}^{\infty} f_n$ to converge uniformly on $X$.

(c) Suppose $X = [a, b]$, all $f_n$ are continuous on $X$ and $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$. Prove that
\[ \int_a^b \sum_{n=1}^{\infty} f_n(x) \, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x) \, dx. \]

Section B. Complex Analysis

7. Exhibit the real and imaginary parts of the function $f(z) = ze^z$ and then verify one of the Cauchy-Riemann equations for $f$ at an arbitrary point $z$.

8. Use the method of residues to find the value of the integral $\int_{-\infty}^{\infty} \frac{x - 1}{x^5 - 1} \, dx$, where the singularity at $0$ is removed from the integrand. (Simplify your answer.)

9. Find the Laurent series in powers of $z$ (i.e., centered at $0$) for the function $f(z) = \frac{z + 9}{(z + 2)(z - 5)}$ that converges in a neighborhood of $z = 3$, and explain briefly what the domain of convergence of your series is.

10. Suppose $f(z)$ is an entire function that satisfies the inequality $|f(z)| \leq |\text{Re} \, z|$, for all $z \in \mathbb{C}$. Prove that $f$ is identically zero on $\mathbb{C}$.

11. Let $f(t)$ be a continuous, real valued function on the interval $[-\pi, \pi]$. For each complex number $z$ define
\[ g(z) = \int_{-\pi}^{\pi} f(t) \sin zt \, dt. \]

(a) Show using Morera’s Theorem that $g(z)$ is an entire function.

(b) Exhibit a power series for $g$ expanded about $0$, with infinite radius of convergence.

12. (a) Find all of the singularities of the function $f(z) = \frac{1}{\sin(e^z)}$.

(b) Find the residue of $f(z)$ at one of its singularities in $\mathbb{C}$ (of your choosing), and explain whether this singularity is a pole or essential.