REAL AND COMPLEX VARIABLES PH.D. QUALIFYING EXAM

May 8, 2012

A passing paper consists of a total of six problems done completely correctly, or five problems done correctly with substantial progress on two others. At least three problems from each of Section A (Real Analysis) and Section B (Complex Analysis) must count toward the passing criteria, and two of these from each section must be completely correct.

Section A. Real Analysis

1. Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{a}{b^2} & \text{if } x = \dfrac{a}{b}, \text{ a rational number in lowest terms,} \\ 0 & \text{otherwise.} \end{cases}$$

   Explain why f is Riemann integrable, and find $\int_0^1 f$.

2. Find with justification the limit: $\displaystyle \lim_{n\to\infty} \int_1^\infty e^{-nt} t^n \, dt.$

3. Let (X, Λ, μ) be a measure space, where μ is a measure on the σ-algebra Λ. A family $\mathcal{F}$ of real valued, measurable functions on X is called *uniformly integrable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{for every } E \in \Lambda \text{ with } \mu(E) < \delta \text{ we have } \quad \int_X |\, f \,| \chi_E \, d\mu < \epsilon, \qquad \text{for all } f \in \mathcal{F}$$

   where χ_E is the characteristic function of E (i.e., this δ works *uniformly* for all $f \in \mathcal{F}$). For any p with $1 < p \le \infty$, prove that the family of all measurable, real valued functions f on X that satisfy $\|\, f \,\|_p \le 1$ is a uniformly integrable family (i.e., the closed unit ball centered at 0 in $L^p(X)$ is uniformly integrable).

4. Let (X, Λ, μ) be a measure space, where μ is a measure on the σ-algebra Λ. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions on X. Prove that the set

$$\{x \in X \mid \{f_n(x)\}_{n=1}^\infty \text{ is a bounded sequence}\}$$

   is a measurable subset of X.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x,y) = (x^3 - 3xy^2, 3x^2y - y^3).$$

   (a) Find the Jacobian matrix of f at an arbitrary point of $\mathbb{R}^2$.

   (b) Find all points (a,b) in the domain of f where f has a local inverse. (Justify your answer.)

6. Let X be a subset of the real line and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real valued functions on X.

   (a) Give the definition of what it means for the sequence $\{f_n\}_{n=1}^{\infty}$ to converge *uniformly* on X.

   (b) Give the definition of what it means for the series $\sum_{n=1}^{\infty} f_n$ to converge *uniformly* on X.

   (c) Suppose $X = [a, b]$, all f_n are continuous on X and $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$. Prove that
$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx.$$

Section B. Complex Analysis

7. Exhibit the real and imaginary parts of the function $f(z) = ze^z$ and then verify *one* of the Cauchy-Riemann equations for f at an arbitrary point z.

8. Use the method of residues to find the value of the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x-1}{x^5-1}\, dx$, where the singularity at 0 is removed from the integrand. (Simplify your answer.)

9. Find the Laurent series in powers of z (i.e., centered at 0) for the function $f(z) = \dfrac{z+9}{(z+2)(z-5)}$ that converges in a neighborhood of $z = 3$, and explain briefly what the domain of convergence of your series is.

10. Suppose $f(z)$ is an entire function that satisfies the inequality $|\,f(z)\,| \le |\,\mathrm{Re}\, z\,|$, for all $z \in \mathbb{C}$. Prove that f is identically zero on $\mathbb{C}$.

11. Let $f(t)$ be a continuous, real valued function on the interval $[-\pi, \pi]$. For each complex number z define
$$g(z) = \int_{-\pi}^{\pi} f(t) \sin zt\, dt.$$

    (a) Show using Morera's Theorem that $g(z)$ is an entire function.

    (b) Exhibit a power series for g expanded about 0, with infinite radius of convergence.

12. (a) Find *all* of the singularities of the function $f(z) = \dfrac{1}{\sin(e^z)}$.

    (b) Find the residue of $f(z)$ at *one* of its singularities in $\mathbb{C}$ (of your choosing), and explain whether this singularity is a pole or essential.