REAL AND COMPLEX ANALYSIS PHD QUALIFYING EXAM

September 19, 2009

The test has two sections, covering real and complex analysis. In order to pass, you must do at least 2 problems from each section **completely correctly**, and you must do a total of 6 problems completely correctly, or 5 completely correctly with substantial progress on 2 others. Some problems have more than one part (e.g., problem 1 in Section I consists of 1a), 1b), and 1c)).

I. REAL ANALYSIS.

1. Let (X, d) be a metric space. Show that, if $\{x_n\}$ is a sequence in X and $p \in X$, then $x_n \to p$ if and only if every subsequence from $\{x_n\}$ has itself a subsequence that converges to p.

2a) Suppose that $f : \mathbf{R} \mapsto \mathbf{R}$ is differentiable everywhere, and that

$$\lim_{x \to \infty} f'(x) = 0.$$

Show that

$$\lim_{x \to \infty} \frac{f(x)}{x} = 0.$$

2b) Use 2a) to prove the following: If $f : \mathbf{R} \mapsto \mathbf{R}$ is differentiable everywhere, and

$$\lim_{x \to \infty} f'(x) = A,$$

where A is a real number, then

$$\lim_{x \to \infty} \frac{f(x)}{x} = A.$$

3. Let $\{E_n\}$ be a sequence of Lebesgue measurable subsets of $\mathbf{R}$ with the property that, for all measurable $A \subset \mathbf{R}$ with finite measure,

$$\lim_{n \to \infty} m(A \setminus E_n) = 0, \tag{1}$$

where $m(\cdot)$ denotes Lebesgue measure. Show that, if f is any Lebesgue integrable function, then

$$\lim_{n \to \infty} \int_{\mathbf{R}} |f \chi_{E_n} - f| \, dm(x) = 0. \tag{2}$$

Conversely, show that if (2) holds for all Lebesgue integrable f, then (1) holds for all A with finite measure.

4. Consider $f(x, y, z) \equiv x^2 - z$ and $g(x, y, z) \equiv x^2 + y^2 - z^2$, both mapping from $\mathbf{R}^3$ into $\mathbf{R}$, and set $S \equiv \{(x, y, z) : \ f = g = 0\}$. We can write $S = P_0 \cup P_+ \cup P_-$, where

$P_0 = \{(0,0,0)\}$, $P_+ = \{(x,y,z) \in S : \ x > 0\}$, and $P_- = \{(x,y,z) \in S : \ x < 0\}$. *This question only deals with* P_+. Find the point or points (x_0, y_0, z_0) on P_+ where the Implicit Function Theorem does *not* guarantee the existence of differentiable functions g_1 and g_2, defined on an open interval I containing z_0, such that $(g_1(t), g_2(t), t) \in P_+$ for all $t \in I$.

5. Exhibit an explicit $f \in L^2([0,1])$ such that f does not belong to $L^p([0,1])$ for any $p \neq 2$. (All L^p spaces are defined with respect to the usual Lebesgue measure.)

6. Find, using the appropriate limit theorem or theorems,

$$\lim_{n\to\infty} \int_0^n \left(\sum_0^n \frac{x^k}{k!} \right) e^{-(3/2)x} \, dm(x).$$

II. COMPLEX ANALYSIS.

In this section, D always denotes the set $\{z \in \mathbf{C} : \ |z| < 1\}$.

1. Use residues to show that, for all $0 < a < 1$,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \, dx = \frac{\pi}{\sin(\pi a)}.$$

2. Suppose $f : \overline{D} \mapsto \mathbf{C}$ is continuous, f is analytic on D, and $|f(z)| < 1$ on the boundary of D. Show that there is a unique $\zeta \in D$ such that $f(\zeta) = \zeta$.

3. Suppose that $f : \mathbf{C} \setminus \{0\} \mapsto \mathbf{C}$ is analytic and, for all $z \neq 0$,

$$|f(z)| \leq \sqrt{|z|} + \frac{1}{\sqrt{|z|}}.$$

Show that f is constant, but that this constant is NOT unique; i.e., that more than one constant function fills the bill.

4. Show that if $u : \mathbf{R}^2 \mapsto \mathbf{R}$ is harmonic, i.e.,

$$u_{xx} + u_{yy} = 0,$$

everywhere, and always positive, then u is constant. You don't need to prove this from scratch, but you must cite the results you use from complex analysis.

5. Let

$$f(z) = \frac{z}{z^2 - 2z - 8}.$$

This function has a Laurent series expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n,$$

valid for all z in the annulus $\{z \in \mathbf{C} : \ 2 < |z| < 4\}$. Compute the coefficients c_n.

6. Suppose that $f : \mathbf{C} \mapsto \mathbf{C}$ is entire and, for all $z \in \mathbf{C}$,

$$f(z) = f(z+1+i) = f(z+2+i).$$

Show that f is constant.