REAL AND COMPLEX ANALYSIS PHD QUALIFYING EXAM

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The test has two sections, covering real and complex analysis. In order to pass, you must do at least 2 problems from each section **completely correctly**, and you must do a total of 6 problems completely correctly, or 5 completely correctly with substantial progress on 2 others. Some problems have more than one part (e.g., problem 1 in Section I consists of 1a), 1b), and 1c)).

I. REAL ANALYSIS.

1a) State what it means for a function $f : [a, b] \mapsto \mathbf{R}$ to be Riemann integrable on $[a, b]$.
1b) State what it means for a sequence of functions $f_n : [a, b] \mapsto \mathbf{R}$ to converge uniformly to a function $f : [a, b] \mapsto \mathbf{R}$.
1c) Let $f_n : [a, b] \mapsto \mathbf{R}$ be a sequence of functions converging uniformly to some $f : [a, b] \mapsto \mathbf{R}$, and suppose that each f_n is Riemann integrable on $[a, b]$. Show that f is Riemann integrable on $[a, b]$ and that

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

2. Suppose that $f : \mathbf{R} \mapsto \mathbf{R}$ satisfies $|f(x) - f(y)| \le |x - y|^{5/4}$ for all x and y in $\mathbf{R}$. Show that f is constant.

3. Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions defined on $\mathbf{R}$, and that, for all n,

$$\int_{\mathbf{R}} |f_n(x)|^2\, dm(x) \le 1,$$

where we are using m to denote Lebesgue measure. Show that, for all $\epsilon > 0$, there is a $\delta > 0$ such that, if $E \subset \mathbf{R}$ is any Lebesgue measurable set satisfying $m(E) < \delta$, then

$$\int_E |f_n(x)|\, dm(x) < \epsilon.$$

4a) Find, with justification, the radius of convergence of the power series:

$$\sum_0^\infty 2^{-\sqrt{n}} x^n.$$

4b) Find, with justification, the radius of convergence of the power series:

$$\sum_0^\infty \frac{x^n}{1 + 2 + 4 + \cdots + 2^n}.$$

You may use without proof the standard tests (ratio, root, etc.) for computing radii of convergence, as well as the values of limits taught in elementary calculus: e.g., that $n^{-1} \log n \to 0$ as $n \to \infty$.

5. Consider the two functions $f_1(x, y, z) \equiv x^2 - 2x + y^2$ and $f_2(x, y, z) \equiv x^2 + y^2 + z^2 - 4$, each mapping from $\mathbf{R}^3$ into $\mathbf{R}$. (Note that f_1 does *not* depend on z.) Define

$$\Sigma \equiv \{(x, y, z) \in \mathbf{R}^3 : \; f_1(x, y, z) = f_2(x, y, z) = 0\}.$$

At what points (x', y', z') on Σ does the Implicit Function Theorem *not* guarantee the existence of an open neighborhood U of z' and differentiable functions $g : U \mapsto \mathbf{R}$ and $h : U \mapsto \mathbf{R}$ such that $(g(z'), h(z'), z') \in \Sigma$ for all $z' \in U$? You do not need to sketch Σ, but it will probably help you to do so.

6. Find, using the appropriate limit theorem or theorems,

$$\lim_{n\to\infty} \int_0^\infty \frac{x^{1/n}}{(e^x + \frac{x}{n})} \, dm(x).$$

II. COMPLEX ANALYSIS.

In this section, D always denotes the set $\{z \in \mathbf{C} : \; |z| < 1\}$.

1. Use residues to show that, for all $a \geq 0$,

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1 + x^2} \, dx = \pi e^{-a}.$$

2. How many zeroes, counting multiplicities, does $f(z) \equiv 3z^2 + e^z$ have inside D?

3a) State and sketch a proof of Morera's Theorem.
3b) Let $f : D \mapsto \mathbf{C}$ be continuous on all of D, and suppose that f is analytic on $D \setminus \{0\}$. Use Morera's Theorem and Cauchy's Theorem (don't prove Cauchy's Theorem) to show that f is analytic on all of D.

4. Consider the function $u(x, y) \equiv e^x \cos y + x^3 - 3xy^2$, which maps from $\mathbf{R}^2$ into $\mathbf{R}$. Show that u is harmonic on $\mathbf{R}^2$, and find a harmonic $v : \mathbf{R}^2 \mapsto \mathbf{R}$ such that

$$f(z) \equiv f(x + iy) = u(x, y) + iv(x, y)$$

is analytic on all of $\mathbf{C}$.

5. Let

$$f(z) = \frac{z}{z^2 - z - 2}.$$

This function has a Laurent series expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n, \tag{1}$$

valid for all z in the annulus $\{z \in \mathbf{C} : \ 1 < |z| < 2\}$. Compute the coefficients c_n. (Hint: Begin by doing a partial-fractions decomposition of $f(z)$. The inner and outer radii of the annulus of convergence will play a very important role here.)

6. Let $f : \mathbf{C} \mapsto \mathbf{C}$ be entire, and suppose that $|f(z)| \leq 13(1 + |z|)^{5.3}$ for all $z \in \mathbf{C}$. Show that f is a polynomial of degree no larger than 5.