

# REAL ANALYSIS PHD QUALIFYING EXAM

September 27, 2008

The test has 10 questions. To pass you must do 7 problems completely correctly or do 6 completely correctly and show substantial progress on 2 others.

1. Let  $\{a_k\}_1^\infty$  be a sequence of real numbers such that  $\sum_1^\infty |a_k| < \infty$ . Set  $s = \sum_1^\infty a_k$ . (You don't need to show that  $s$  exists.) Show that the value of this sum does not depend on the order in which the  $a_k$ 's are added. In other words, show that if  $\sigma : \mathbf{N} \mapsto \mathbf{N}$  is a one-to-one mapping of  $\mathbf{N}$  (the natural numbers) onto itself, and we set  $b_k = a_{\sigma(k)}$ , then  $\sum_1^\infty b_k$  also equals  $s$ .

2. Let  $\{(a_k, b_k)\}_1^\infty$  be a collection of intervals contained in  $[0, 1]$ , and suppose that

$$\sum_1^\infty (b_k - a_k) > 1.$$

Show that the set of intervals  $\{(a_k, b_k)\}$  cannot be pairwise disjoint.

3. Let  $(X, \mathcal{M}, \mu)$  be a measure space for which  $0 < \mu(X) < \infty$ , and suppose that  $f \in L^\infty(X, \mathcal{M}, \mu)$ . Show that  $f \in L^p(X, \mathcal{M}, \mu)$  for all  $p < \infty$ , and also show that

$$\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty.$$

4. Let  $f : [0, 1] \mapsto \mathbf{R}$  be continuous (with respect to the usual, absolute-value metric) and one-to-one, and suppose that  $f(0) < f(1)$ . Show that  $f$  is strictly increasing on  $[0, 1]$ ; i.e., show that, for all  $x$  and  $y$  in  $[0, 1]$ ,  $x < y$  implies  $f(x) < f(y)$ .

5. Show that, if  $(X, \mathcal{M}, \mu)$  is any measure space, the following two statements, regarding sequences of sets  $\{E_k\} \subset \mathcal{M}$ , are *equivalent*: a) For all  $A \in \mathcal{M}$  such that  $\mu(A) < \infty$ ,

$$\lim_{k \rightarrow \infty} \mu(A \setminus E_k) = 0;$$

b) For all  $f \in L^1(X, \mathcal{M}, \mu)$ ,  $f\chi_{E_k} \rightarrow f$  in  $L^1$  as  $k \rightarrow \infty$ . (Hint: Begin by showing that, if a) holds, b) is true for all integrable simple functions  $f$ .)

6. Let  $(M, d)$  be a metric space. Show that if  $\{x_k\} \subset M$  is any Cauchy sequence, and some subsequence  $\{x_{n_k}\}$  converges to a point  $p \in M$ , then the whole sequence  $\{x_k\}$  converges to  $p$ .

7. If  $E \subset \mathbf{R}^d$ , a point  $x \in \mathbf{R}^d$  is called a *condensation point* of  $E$  if, for all  $r > 0$ ,  $B(x; r) \cap E$  is uncountable, where  $B(x; r) \equiv \{y \in \mathbf{R}^d : \|x - y\| < r\}$  and  $\|\cdot\|$  is the usual Euclidean norm. (Notice that  $x$  need not belong to  $E$ .) Show that, for any  $E \subset \mathbf{R}^d$ , the set of  $E$ 's condensation points is closed.

8. Let  $(X, \mathcal{M}, \mu)$  be a measure space, and suppose that  $\{f_n\}$  is a sequence from  $L^+(X, \mathcal{M}, \mu)$ , the family of non-negative measurable functions. Show that, if  $f_n \rightarrow f \in L^+$  pointwise, and

$$\lim_{n \rightarrow \infty} \int_X f_n(x) d\mu(x) = \int_X f(x) d\mu(x) < \infty,$$

then, for all  $E \in \mathcal{M}$ ,

$$\lim_{n \rightarrow \infty} \int_E f_n(x) d\mu(x) = \int_E f(x) d\mu(x).$$

You may use standard limit theorems (Fatou, Monotone Convergence, etc.) without proof.

9. Consider the two surfaces in  $\mathbf{R}^3$ :

$$\Sigma_1 \equiv \{(x, y, z) : x \sin z - y \cos z = 0\};$$

$$\Sigma_2 \equiv \{(x, y, z) : x^2 + 4y^2 = 1\},$$

and let  $\Gamma = \Sigma_1 \cap \Sigma_2$ . Show that, for every  $(x_0, y_0, z_0) \in \Gamma$ , the Implicit Function Theorem implies the existence of a differentiable, one-to-one  $\phi(t) = (\phi_1(t), \phi_2(t))$ , defined on some non-trivial open interval  $I = (z_0 - \delta, z_0 + \delta)$ , and mapping into  $\mathbf{R}^2$ , satisfying  $\phi(z_0) = (x_0, y_0)$  and such that  $(\phi_1(t), \phi_2(t), t) \in \Gamma$  for all  $t \in I$ .

10. Let  $\mathcal{M}$  be the  $\sigma$ -algebra of subsets of  $\mathbf{R}$  that are countable or have countable complements. Define  $\mu : \mathcal{M} \mapsto \{0, 1\}$  by:

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable;} \\ 1 & \text{if } \mathbf{R} \setminus E \text{ is countable.} \end{cases} \quad (1)$$

Let  $\mu^* : \mathcal{P}(\mathbf{R}) \mapsto [0, \infty]$  be the outer measure induced by  $\mu$ . a) Find an expression for  $\mu^*$ , analogous to (1), valid for all  $E \subset \mathbf{R}$ . b) Let  $\mathcal{M}^*$  be the  $\sigma$ -algebra of  $\mu^*$ -measurable sets. Show that  $\mathcal{M}^* = \mathcal{M}$ .