REAL ANALYSIS PHD QUALIFYING EXAM

September 20, 2007

A passing grade is 6 problems done completely correctly, or 5 done completely correctly with substantial progress on 2 others.

1. Let (X, d) be a compact metric space, where we take "compact" to mean "every open cover of X has a finite subcover." Show that every sequence $\{x_n\}_1^\infty$ in X has a subsequence converging to some $z \in X$.

2a). Let $\{b_n\}$ be a sequence of positive numbers which is *unbounded*: $\sup_n b_n = \infty$. Show that there is a sequence of positive numbers $\{a_n\}$ such that $\sum a_n < \infty$, but for which $\sum a_n b_n = \infty$.
2b) Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n = \infty$. Show that there is a sequence of positive numbers $\{d_n\}$ such that $\lim_n d_n = 0$, but for which $\sum c_n d_n = \infty$.

3. Prove the following: If f is differentiable on $(0, 1)$ and $f'(1/4) < 0 < f'(3/4)$, there is a $c \in (1/4, 3/4)$ such that $f'(c) = 0$.

4. Let $\{f_n\}$ be a sequence of functions in $L^p(\mathbf{R}, \mathcal{L}, m)$, where $1 < p < \infty$, $\mathcal{L}$ is the Lebesgue measurable sets, and m denotes Lebesgue measure. Suppose that

$$\sup_n \|f_n\|_p < \infty. \tag{1}$$

Show that $\{f_n\}$ is *uniformly integrable*, which means: for every $\epsilon > 0$ there is a $\delta > 0$ such that, for all $E \in \mathcal{L}$, $m(E) < \delta$ implies

$$\sup_n \int_E |f_n| \, dm < \epsilon.$$

Also, give an example of a sequence in L^1 satisfying (1) for $p = 1$, but which is not uniformly integrable.

5. Let f and g belong to $L^2(\mathbf{R}, \mathcal{L}, m)$. Show that

$$\lim_{n\to\infty} \int f(x)\, g(x+n)\, dm = 0.$$

6. Let $\|\cdot\|$ denote the usual Euclidean norm in $\mathbf{R}^d$. Let A and B be non-empty subsets of $\mathbf{R}^d$, where A is compact and B is closed (with respect to the usual topology). Show that there exist points $a \in A$ and $b \in B$ such that, for all $x \in A$ and $y \in B$,

$$\|a - b\| \leq \|x - y\|.$$

Show that such points need not exist if A is merely assumed to be closed.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose that $\{E_n\}_1^\infty$ is a sequence from $\mathcal{M}$ with the property that

$$\lim_{n\to\infty} \mu(X \setminus E_n) = 0,$$

Let $G \subset X$ be the set of x's that belong to only finitely many of the sets E_n; i.e., $x \in G$ if and only if x belongs to at most finitely many E_n's. Show that $G \in \mathcal{M}$ and $\mu(G) = 0$.

8. Show that, if $f \in L^1(\mathbf{R}, \mathcal{L}, m)$,

$$\lim_{n\to\infty} \int f(x)\,(\sin(nx))^2\,dm = (1/2) \int f(x)\,dm.$$

9. Suppose that $\{f_n\}$ is a sequence in $L^2(\mathbf{R}, \mathcal{L}, m)$ such that $\sum_1^\infty \|f_n\|_2 < \infty$ and $\sum_1^\infty f_n(x) = 0$ for (Lebesgue-)almost-every $x \in \mathbf{R}$. Prove that, for all $g \in L^2(\mathbf{R}, \mathcal{L}, m)$,

$$\sum_1^\infty \int f_n(x)\,g(x)\,dm$$

exists and equals 0.

10. Define $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ by $f(x, y) = (x^2 - y, x^4 + y^2)$, and let $(a, b) \in \{(x, y) : \ x < 0,\ y > 0\}$. Show that f is one-to-one on some open set U containing (a, b), and that there is a differentiable $g : f[U] \mapsto U$ such that $f(g(x, y)) = (x, y)$ for all $(x, y) \in U$. In other words, prove that, at every point in the open second quadrant, f has a locally defined differentiable inverse.