A passing paper consists of 7 problems solved completely, or 6 solved completely with substantial progress on 2 others.

1. Let \((X, d)\) be a metric space. A set \(E \subseteq X\) is called discrete if there is \(\delta > 0\) such that, for all \(x, y\) in \(E\) with \(x \neq y\) we have \(d(x, y) > \delta\). Show that a discrete set is necessarily closed. (Use any standard definition of “closed set” in a metric space.)

2. Suppose that \(f : (0, 1) \to \mathbb{R}\) is differentiable on all of \((0, 1)\) and \(f'(1/4) < 0 < f'(3/4)\). Show that there is a \(c \in (1/4, 3/4)\) such that \(f'(c) = 0\).

3. Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is differentiable on all of \(\mathbb{R}\) and \(\lim_{x \to \infty} f'(x) = A\), where \(A\) is a real number. Show that \(\lim_{x \to \infty} \frac{f(x)}{x}\) exists and equals \(A\). [Hint: Show this for \(A = 0\) first.]

4. Let \(f : [1, \infty) \to [0, \infty)\) be a non-increasing function. Prove that
\[
\int_1^\infty f(x) \, dx < \infty \quad \text{if and only if} \quad \sum_{k=0}^\infty 2^k f(2^k) < \infty.
\]

5. Consider the two surfaces in \(\mathbb{R}^3\),
\[
\Sigma_1 = \{(x, y, z) : z = xy\}
\]
\[
\Sigma_2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\},
\]
and let \(\Gamma \equiv \Sigma_1 \cap \Sigma_2\). For almost all of the points \((\tilde{x}, \tilde{y}, \tilde{z}) \in \Gamma\), the Implicit Function Theorem guarantees the existence of a differentiable function \(g = (g_1, g_2)\), defined on some open neighborhood \(U\) of \(\tilde{z}\) and mapping into \(\mathbb{R}^2\), such that \((g_1(z), g_2(z), z) \in \Gamma\) for all \(z \in U\). But there are FOUR points \((\tilde{x}, \tilde{y}, \tilde{z})\) where the IFT does not guarantee the existence of such a \(g\). Find the points, with justification.

6. Let \((X, \mathcal{M}, \mu)\) be a measure space, where \(\mathcal{M}\) is a \(\sigma\)-algebra, and let \(g : X \to [0, \infty]\) be a non-negative measurable function. For each \(E \in \mathcal{M}\) define
\[
\nu(E) = \int g \chi_E \, d\mu.
\]

(a) Show that \(\nu(E)\) defines a measure on \(\mathcal{M}\). (You may quote without proof any standard theorems from measure theory in your argument.)

(b) In a similar fashion, show that if \(f : X \to [0, \infty]\) is any non-negative measurable function, then \(\int f \, d\nu = \int f \, g \, d\mu\).

7. Suppose that \((X, \mathcal{M}, \mu)\) is a finite measure space, and \(\{E_k\}\) is a sequence of sets from \(\mathcal{M}\) such that \(\mu(E_k) > 1/100\) for all \(k\). Let \(F\) be the set of points \(x \in X\) which belong to infinitely many of the sets \(E_k\).
(a) Show that $F \in \mathcal{M}$, i.e., $F$ is a measurable set.

(b) Prove that $\mu(F) \geq 1/100$.

(c) Give an example to show that conclusion (b) can fail if $\mu(X) = \infty$.

8. Find the value of

$$\lim_{n \to \infty} \int_0^\infty \left( \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x} \, dx,$$

and justify your assertion by quoting appropriate facts from calculus and one or more limit theorems from measure theory.

9. Let $(X, || \ ||)$ be a normed linear space.

(a) State what it means for $(X, || \ ||)$ to be a Banach space, and give an example, with details, of a normed linear space that is not a Banach space.

(b) Let $\{x_k\}_{k=1}^\infty$ be a sequence in $X$ and let

$$S_N = \sum_{k=1}^{N} x_k$$

be the usual $N$th partial sum of the series $\sum_{k=1}^{\infty} x_k$. The series is said to be summable if the sequence $\{S_N\}_{N=1}^\infty$ of partial sums converges to an element of $X$. The series is called absolutely summable if $\sum_{k=1}^{\infty} ||x_k|| < \infty$.

Prove that $(X, || \ ||)$ is a Banach space if and only if every absolutely summable series is summable. (You may use without proof the fact that if a Cauchy sequence has a subsequence that converges to $L$, then the entire sequence also converges to $L$.)

10. Let $\phi \in L^\infty(\mathbb{R})$ (the measure on $\mathbb{R}$ is the usual Lebesgue measure). Show that

$$\lim_{n \to \infty} \left( \int_\mathbb{R} \frac{\left| \phi(x) \right|^n}{1 + x^2} \, dx \right)^{1/n}$$

exists and equals $\| \phi \|_{\infty}$. 