REAL ANALYSIS PH.D. QUALIFYING EXAM

January 31, 2009

A passing paper consists of 7 problems solved completely, or 6 solved completely with substantial progress on 2 others.

1. Let (X, d) be a metric space. A set $E \subseteq X$ is called *discrete* if there is $\delta > 0$ such that, for all x and y in E with $x \neq y$ we have $d(x, y) > \delta$. Show that a discrete set is necessarily closed. (Use any standard definition of "closed set" in a metric space.)

2. Suppose that $f : (0, 1) \to \mathbb{R}$ is differentiable on all of $(0, 1)$ and $f'(1/4) < 0 < f'(3/4)$. Show that there is a $c \in (1/4, 3/4)$ such that $f'(c) = 0$.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on all of $\mathbb{R}$ and $\lim_{x\to\infty} f'(x) = A$, where A is a real number. Show that $\lim_{x\to\infty} \frac{f(x)}{x}$ exists and equals A. [Hint: Show this for $A = 0$ first.]

4. Let $f : [1, \infty) \to [0, \infty)$ be a non-increasing function. Prove that

$$\int_1^\infty f(x)\,dx < \infty \qquad \text{if and only if} \qquad \sum_{k=0}^{\infty} 2^k f(2^k) < \infty.$$

5. Consider the two surfaces in $\mathbb{R}^3$,

$$\Sigma_1 = \{(x, y, z) : \ z = xy\}$$
$$\Sigma_2 = \{(x, y, z) : \ x^2 + y^2 + z^2 = 1\},$$

and let $\Gamma \equiv \Sigma_1 \cap \Sigma_2$. For almost all of the points $(\tilde{x}, \tilde{y}, \tilde{z}) \in \Gamma$, the Implicit Function Theorem guarantees the existence of a differentiable function $g = (g_1, g_2)$, defined on some open neighborhood U of $\tilde{z}$ and mapping into $\mathbb{R}^2$, such that $(g_1(z), g_2(z), z) \in \Gamma$ for all $z \in U$. But there are FOUR points $(\tilde{x}, \tilde{y}, \tilde{z})$ where the IFT does not guarantee the existence of such a g. Find the points, with justification.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space, where $\mathcal{M}$ is a σ-algebra, and let $g : X \to [0, \infty]$ be a non-negative measurable function. For each $E \in \mathcal{M}$ define

$$\nu(E) = \int g\,\chi_E\,d\mu.$$

(a) Show that $\nu(E)$ defines a measure on $\mathcal{M}$. (You may quote without proof any standard theorems from measure theory in your argument.)

(b) In a similar fashion, show that if $f : X \to [0, \infty]$ is any non-negative measurable function, then $\int f\,d\nu = \int f\,g\,d\mu$.

7. Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space, and $\{E_k\}$ is a sequence of sets from $\mathcal{M}$ such that $\mu(E_k) > 1/100$ for all k. Let F be the set of points $x \in X$ which belong to infinitely many of the sets E_k.

(a) Show that $F \in \mathcal{M}$, i.e., F is a measurable set.

(b) Prove that $\mu(F) \geq 1/100$.

(c) Give an example to show that conclusion (b) can fail if $\mu(X) = \infty$.

8. Find the value of

$$\lim_{n\to\infty} \int_0^\infty \left(\sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x}\, dx,$$

and justify your assertion by quoting appropriate facts from calculus and one or more limit theorems from measure theory.

9. Let $(X, ||\quad||)$ be a normed linear space.

(a) State what it means for $(X, ||\quad||)$ to be a Banach space, and give an example, with details, of a normed linear space that is **not** a Banach space.

(b) Let $\{x_k\}_{k=1}^\infty$ be a sequence in X and let

$$S_N = \sum_{k=1}^{N} x_k$$

be the usual N^{th} partial sum of the series $\sum\limits_{k=1}^{\infty} x_k$. The series is said to be *summable* if the sequence $\{S_N\}_{N=1}^\infty$ of partial sums converges to an element of X. The series is called *absolutely summable* if if $\sum\limits_{k=1}^{\infty} ||\, x_k \,|| < \infty$.
Prove that $(X, ||\quad||)$ is a Banach space if and only if every absolutely summable series is summable. (You may use without proof the fact that if a Cauchy sequence has a subsequence that converges to L, then the entire sequence also converges to L.)

10. Let $\phi \in L^\infty(\mathbb{R})$ (the measure on $\mathbb{R}$ is the usual Lebesgue measure). Show that

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}} \frac{|\phi(x)|^n}{1+x^2}\, dx \right)^{1/n}$$

exists and equals $\|\phi\|_\infty$.