Ph.D. QUALIFYING EXAM IN REAL ANALYSIS

January 10, 2008
Three hours

There are 11 questions. A passing paper consists of 6 questions done completely correctly, or 5 questions done correctly with substantial progress on 2 others.

1. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Assume that every convergent subsequence converges to the same real number. Prove that there is a real number L such that the entire sequence converges to L.
[Note: The hypotheses allow the possibility that $\{x_n\}_{n=1}^{\infty}$ has *no* convergent subsequences, so your proof must subsume this case.]

2. Let f be a real valued function that is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Assume $f(0) = 0$ and $|\, f'(x)\,| \le |\, f(x)\,|$ for all $x \in (0, 1)$. Prove that $f = 0$ on $[0, 1]$.

3. Let $f_n(x) = \dfrac{1}{1 + x^{2n}}$ for $x \in \mathbb{R}$.

(a) Find where $\{f_n\}_{n=1}^{\infty}$ converges pointwise, and describe the limit function $F(x)$.

(b) Describe the intervals in $\mathbb{R}$ on which $\{f_n\}_{n=1}^{\infty}$ converges uniformly to F.

4. Suppose f, g and h are bounded real valued functions on $[0, 1]$ with $f \le g \le h$. If f and h are Riemann integrable with $\int_0^1 f = \int_0^1 h$, prove that g is Riemann integrable.

5. Let $F(x, y, z) = (x^2 + z^2 - 4)^2 + xy - 2008$. Find all points (x, y, z) such that the Implicit Function Theorem does *not* provide a local implicit function $f(x, y) = z$ such that $F(x, y, f(x, y)) = 0$. Describe this set geometrically as a subset of $\mathbb{R}^3$ (eg., a sphere, cone, etc.).

6. Let $g : (0, \infty) \longrightarrow \mathbb{R}$ by

$$g(x) = n \quad \text{when} \quad x \in ((n-1)^2, n^2], \qquad \text{for each } n \in \mathbb{N}.$$

Since g is increasing and left continuous let μ be the Lebesgue-Stieltjes measure obtained from g on the semiring of half open intervals:

$$\mu([a, b)) = g(b) - g(a), \qquad \text{for all } b \ge a > 0.$$

Prove that every subset of $(0, \infty)$ is μ^*-measurable, and find a simple expression for $\mu^*(E)$ for an arbitrary subset E of $(0, \infty)$.
[You may assume μ is a measure on this semiring, and μ^*, its Carathéodory extension, is an outer measure; you may quote basic results from measure theory without proof.]

7. Let λ denote Lebesgue measure on the real line.

(a) Prove that there is an open set $\mathcal{O}$ that is dense in $\mathbb{R}$ with $\lambda(\mathcal{O}) < 1$.

(b) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} - \mathcal{O}$ is uncountable.

(c) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} - \mathcal{O}$ is not compact.

8. (a) State the Lebesgue Dominated Convergence Theorem for $f_n : [0,1] \longrightarrow \mathbb{R}$.

(b) Use (a) to evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{dx}{(1+\frac{x}{n})^n \, x^{\frac{1}{n}}}$$

where dx denotes integration with respect to Lebesgue measure.
[Be sure to explain why the hypotheses are satisfied when you quote (a).]

9. Either prove or provide an explicit counterexample to each of the following assertions: (you may quote without proof familiar relations and containments between L^p-spaces)

(a) If $f, g \in L^2([0,1])$ then $fg \in L^2([0,1])$. (Lebesgue measure)

(b) If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$. (Lebesgue measure)

(c) If $f, g \in \ell^2$ then $fg \in \ell^2$. (counting measure)

10. Assume g is a continuous real valued function on $[-\pi, \pi]$ with $g(-\pi) = g(\pi)$, and suppose that $\int_{-\pi}^{\pi} g(t) \sin nt \, dt = 0$ for all natural numbers n. Prove that g is an even function (i.e., $g(-x) = g(x)$ for all x). [Hint: Consider $g(x) - g(-x)$.]

11. Suppose $\{e_1, e_2, \dots\}$ is an orthonormal basis of $L^2([0,1])$. Extend each e_i to a function on $\mathbb{R}$ by making it zero outside $[0,1]$. For each integer n define $e_{i,n}(x) = e_i(x-n)$ (the translate of $e_i(x)$ by n). Prove that $\{e_{i,n} \mid i \in \mathbb{N},\ n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$. (All L^2-spaces are with respect to Lebesgue measure.)