Numerical Analysis: MATH 237 and 337

Qualifying Examination

Spring 2016

For passing this exam, four problems must be done completely correctly and one problem must be at least 75% correct. Of the four completed problems, two must come from each part of the exam. (E.g., completing problems 1, 2, 3, 4 and doing problem 5 at 75% will not be a passing score; on the other hand, completing 1, 2, 4, 6 and doing either 3 or 5 at 75% will constitute a pass.)

You have three hours to complete the exam.

Note: Make sure to provide **explanations** to all steps of your solutions and to **explain** all answers to posed questions.

Part 1

1. Consider $f(x) = \ln(x+3)$.

   (a) Use $x_0 = 0$, $x_1 = 0.2$ and $x_2 = 0.5$ to construct the Lagrange interpolating polynomial of degree at most two to approximate $f(0.25)$ and find the absolute error of this approximation.

   (b) Use the Theorem on polynomial interpolation error to find an error bound for the above approximation.

2. Find the natural cubic spline for the data points $(1,5)$, $(2,3)$ and $(3,-1)$.

3. (a) Starting from $p_0(x) = 1$, $p_1(x) = x$ and $p_2(x) = x^2$, compute a set of orthogonal polynomials: $q_0(x)$, $q_1(x)$ and $q_2(x)$, subject to the following inner product:

   $$\langle f, g\rangle = \int_{-1}^{1} x^2 f(x) g(x) dx.$$

   Then find the two roots, x_1 and x_2, of $q_2(x)$.

   (b) Determine the coefficients A_1 and A_2 that make the following formula exact for all polynomials of maximum degree m:

   $$\int_{-1}^{1} x^2 f(x) dx \approx A_1 f(x_1) + A_2 f(x_2),$$

   where x_1 and x_2 are found in (a). What is m?

See Part 2 on next page.

Part 2

4. Method

$$Y_{n+1} = \frac{1}{3}(2Y_n + Y_{n-1}) + \frac{h}{3}(f_{n-1} + f_n + 2f_{n+1}),$$

where $f_n = f(x_n, Y_n)$, etc., can be used to solve an ordinary differential equation (ODE) $y' = f(x, y)$ or systems of first-order ODEs.

(a) Show that this method is second-order accurate.

(b) Write out the difference equations that result when this method is applied to integrate the equation of the harmonic oscillator $u'' = -\Omega^2 u$.

5. (a) Propose a 2nd-order accurate discretization of the equation

$$\big(p(x)\, u_x\big)_x = q(x)u + r(x). \tag{1}$$

Explain why it is 2nd-order accurate.

(b) Use this discretization to set up a linear system for the boundary-value problem given by Eq. (1) and by the boundary conditions

$$u_x(0) = \alpha, \qquad u(1) = \beta, \tag{2}$$

where α, β are some given constants.

Specifically, make sure to use a 2nd-order accurate approximation to handle the Neumann boundary condition at $x = 0$. Then, explain how you will determine the unknown solution u at all grid nodes that you have introduced.

6. Consider the Heat equation $u_t = u_{xx}$ on the domain $0 \le x \le 1$ and subject to the Dirichlet boundary conditions. For future reference, a first-order in time, explicit scheme for this equation is:

$$U_m^{n+1} - U_m^n = r\left(U_{m+1}^n - 2U_m^n + U_{m-1}^n\right), \tag{1}$$

where $r = \kappa/h^2$ and κ and h are the step sizes in time and space, respectively. The amplification coefficient, obtained by the von Neumann analysis, of one time step of scheme (1) equals:

$$\rho = 1 - 4r\sin^2\frac{\beta h}{2}, \tag{2}$$

where β is the wavenumber of the Fourier harmonic $\exp[i\beta m h]$. The stability condition for scheme (1) is: $r \le 0.5$. You do *not* need to derive any of the results stated above.

(a) Write out a fully *implicit*, first-order in time scheme for the Heat equation. (Just stating the scheme is enough; you do *not* need to prove that it is first-order accurate.) Use the von Neumann analysis to derive its amplification coefficient of one time step and determine the stability condition for this scheme.

(b) Suppose your initial condition is a discontinuous function of x. Which scheme, (1) or the one you have derived in part (a), will more efficiently smoothen out the discontinuity? Answer this question for each of the two values of r: $r = 0.2$ and $r = 0.4$.