Numerical Analysis: MATH 237 and 337

Qualifying Examination **Spring 2014**

Four problems must be completed, and in addition to those, one more problem must be attempted. **Note**: At least one of questions 6 or 7 must be completed. For the attempted problem, you must correctly outline the main idea of the solution and begin the calculation, but need not have finished.
You have three hours to complete the exam.

1. Perform several iterations (see below) of the secant method to solve equation

$$x^2 - 3x + 2 = 0,$$

using as two initial guesses $x_1 = 3$ and $x_2 = 4$. The error e_n of the secant method has the following asymptotic pattern:

$$e_{n+1} = \alpha\, e_n^G,$$

where $G > 1$ and in this particular case, $\alpha = 1$.

Your task is to determine G to one decimal place (e.g., $G \approx 1.5$). That is, stop iterating when the first decimal place of G stops changing.

Suggestion: To save time on calculations, work out an expression in terms of x_n and x_{n-1} for the divided difference used by the secant method.

2. Find a natural cubic spline function whose knots are -1, 0, 1 and 2, and that takes these values: $(-1, 0)$, $(0, 3)$, $(1, 11)$ and $(2, 24)$.

3. Find a formula of the form

$$\int_0^1 x f(x) dx \approx A_0 f(x_0) + A_1 f(x_1)$$

that is exact for all polynomials of degree ≤ 3.

4. Method

$$Y_{n+1} = Y_{n-1} + \frac{h}{3}(f_{n-1} + 4f_n + f_{n+1}),$$

where $f_n = f(x_n, Y_n)$, etc., can be used to solve the ODE $y' = f(x, y)$ or systems of first-order ODEs.

(a) For a single ODE, show that this method has the global error of order four.

(b) Write out the difference equations that result when this method is applied to integrate the equation of the harmonic oscillator $u'' = -\Omega^2 u$. (You do *not* need to prove anything in this part.)

5. Explain how you would solve the boundary-value problem

$$y'''' = p(x)y' + q(x)y + r(x),$$

$$y(0) = \alpha, \quad y'(0) = \beta, \qquad y'(1) = \gamma, \quad y''(1) = \delta$$

by the shooting method. You should describe and explain all relevant details, but should not solve any equations.

6. Write out a finite difference scheme to solve the nonlinear Heat equation

$$u_t = F(u)u_{xx} + G(u)u_x + H(x,t)$$

with error of the order $O(\kappa^2 + h^2)$. Here $F(u)$, $G(u)$, and $H(x,t)$ are arbitrary smooth functions, assumed to be known. Note that $H(x,t)$ does not depend on u. *Explain* why the order of your scheme is $O(\kappa^2 + h^2)$.

Your scheme may be implicit or semi-implicit (you need to present *only one, not both*).

- If you choose an implicit scheme, describe how you would solve the resulting system of nonlinear equations. Present all relevant details, but *do not actually solve* any equations.
- If you choose a semi-implicit scheme, explain how you would solve for a certain auxiliary variable. After you have solved for that variable, explain how you would complete one time step of your scheme to solve for $u(x, t_{n+1})$.

7. Consider the unidirectional wave equation

$$u_t = c\, u_x,$$

where $c > 0$ is the wave speed. This equation describes propagation of initial disturbances that is *not* accompanied by any dissipation.

Let us denote the finite-difference solution of this equation by: $U_m^n = U(x_m, t_n)$. Let the mesh size be h and the time step be κ. In the numerical scheme in question, the right-hand-side of this equation is discretized by the central difference:

$$u_x \;\rightarrow\; \frac{U_{m+1}^n - U_{m-1}^n}{2h}.$$

(a) Propose a discretization of the left-hand side that is consistent with the nature of this equation, as mentioned in the first paragraph. Give a brief explanation.

(b) In the scheme that you have written for part (a) there is a parameter $\mu = c\kappa/h$. Assume

$$\mu \le 1$$

and show that your scheme is stable under this condition.

(c) Now assume $\mu > 1$. A Fourier harmonic with what value of (βh) will be most unstable for $\mu > 1$? (You should not compute any maxima. Instead, find that most unstable harmonic by looking at a few simple values of (βh).)