Numerical Analysis PhD Qualifying Exam
University of Vermont, Winter 2011

Instructions: *Four problems must be completed, and in addition to those, one more problem must be attempted. At least two problems from 1-4 and at least two problems from 4-7 must be completed.* **Note** *that Question 4 can count towards either section, but not both. To have attempted a problem, you must correctly outline the main idea of the solution and begin the calculation, but need not have finished. You have three hours to complete the exam.*

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1. (a) Given an initial guess x_0, derive Newton's method to find a better guess x_1 for approximating the root of a function $f(x)$. (b) Apply Newton's method to the function $f(x) = 1/x$ using an initial guess of $x_0 = 1$ and find a (simple) analytical expression for x_{50}.

2. Solve the following linear system with naive Gaussian elimination (i.e. without partial pivoting)

$$\begin{bmatrix} \frac{\epsilon_{mach}}{10} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

using (1) infinite precision and (2) a computer whose machine epsilon is given by ϵ_{mach}. Label your solutions $\vec{x}_{true}$ and $\vec{x}_{comp}$ respectively. Why is there such large difference between the two? **Note** that the first pivot $\frac{\epsilon_{mach}}{10}$ is much larger than the smallest number the computer can represent.

3. Apply Gram-Schmidt to find a QR-factorization of the matrix.

$$A = \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$$

4. Given an IVP $y' = f(t, y)$, methods for numerical integration are distinguished by their approximation of the integral in the formula $y(t+h) = y(t) + \int_{t_i}^{t_{i+1}} f(t, y)dt$. Derive the degree-2 Adam's Bashforth method (AB2) given by $w_{i+1} = w_i + \frac{h}{2}(3f_i - f_{i-1})$ in two steps:

   (1) Approximate $f(t, y)$ with a polynomial $P_n(t)$ of degree n interpolating the $n+1$ points $(t_{i-n}, f_{i-n}), \ldots, (t_{i-1}, f_{i-1}), (t_i, f_i)$. **Note** that you should determine the degree n based on the specified order of accuracy of AB2. It may help to label the constant step-size in time $h = t_i - t_{i-1}$.

   (2) Evaluate the integral $\int_{t_i}^{t_{i+1}} P_n(t)dt$

5. Method

$$Y_{n+1} - Y_{n-1} = \frac{h}{8}(5f_{n+1} + 6f_n + 5f_{n-1}), \qquad \text{where} \quad f_n \equiv f(x_n, Y_n), \text{ etc.} \tag{1}$$

   can be used to solve the initial-value problem

$$y' = f(x, y), \qquad y(x_0) = y_0. \tag{2}$$

   Using the equation $y' = -\lambda y$ as a model problem ($\lambda > 0$), show that this method is A-stable.
   Note: A notation $h\lambda/8 \equiv z$, so that $z > 0$, should be helpful.

6. Describe how you would solve a boundary-value problem on $x \in [a,\, b]$:

$$y'' = y^3 - x, \qquad y(a) = \alpha, \quad y(b) = \beta \tag{1}$$

with *second-order accuracy*.

If you choose to use a finite-difference discretization, do the following:

- Write the equation at an internal point.
- Write the equations at the boundary points.
- Write your system of equations in matrix (or matrix-vector) form.
- Describe what method you would use to solve (or attempt to solve) your system of equations. Provide only brief necessary details about the method's setup; do *not* go deeply into its workings. *Note*: If several alternative methods can be used, describe **only one** of them, **not all**. Also, your method does *not* have to be the best one; it should be just a reasonable method.

If you choose to use the shooting method, do the following:

- Write the equation (or equations) that you would be solving numerically.
- Explain what method(s) you would use to solve this equation (or these equations). You do *not* need to write the equations of the method(s); just write its (their) name(s) and, if needed, briefly justify your choice.

Note: You need to describe **just one** of the above methods of solution, **not both**.

7. A method

$$U_j^{n+1} - U_j^n = \frac{\kappa}{h^2}\left(U_{j+1}^n - U_j^n - U_j^{n+1} + U_{j-1}^{n+1}\right) \tag{1}$$

is proposed by some people in the computational finance community in connection with solving the initial-boundary-value problem for the Heat equation:

$$u_t = u_{xx}, \qquad x \in [0,\, 1], \quad t \geq 0; \qquad u(0,t) = \alpha, \quad u(1,t) = \beta, \quad u(x,0) = \varphi(x). \tag{2}$$

(In Eq. (1), κ and h are the temporal and spatial steps, and U_j^n is the numerical approximation to $u(jh,\, n\kappa)$.)

(a) Explain how this seemingly implicit scheme can be solved recursively, i.e. without inverting any matrix.
Hint: Draw the grid for the BVP (2) and try to find the solution node-by-node at the first time level. (The initial condition is prescribed at the zeroth time level.)

(b) Use the von Neumann analysis to show that this scheme is unconditionally stable.