COMPLEX VARIABLES PH.D. QUALIFYING EXAM

May 20, 2008

There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let $\mathbb{D}$ denote the open disc of radius 1 centered at the origin.

1. Let f be holomorphic on a connected open set U. Prove that if $f(z)^2 = \overline{f(z)}$ for all $z \in U$ then f is constant on U. Find all possible values for f.

2. Let γ be the circle of radius 5 centered at 0. Evaluate with brief justification the integrals:

(a) $\displaystyle\int_\gamma \frac{\overline{z}}{z-1}\,dz,$ (b) $\displaystyle\int_\gamma e^{1/z}\,dz.$

3. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point $z = 2$ for the following function (explain briefly how the inner and outer radii of the annulus are determined):

$$f(z) = \frac{z}{1-z^2} + \frac{6}{(z-4)^2}.$$

4. Use the calculus of residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\,dx.$$

5. Prove that if f is entire and there are positive real numbers A, B and k such that $|f(z)| \leq A + B|z^k|$ for all $z \in \mathbb{C}$, then f is a polynomial.

6. Let f be analytic on the closed unit disc $\overline{\mathbb{D}}$, and assume $|f(z)| < 1$ on its boundary. Prove that there is one and only one point $z_0 \in \mathbb{D}$ such that $f(z_0) = z_0$.

7. (a) Exhibit an entire function, $P(z)$, that has simple zeros at the numbers $\sqrt{n}$ for each positive integer n, and no other zeros.

(b) For the function P you gave in part (a), describe P'/P as an infinite series (not necessarily a Taylor series however).

8. Define $f(z) = \displaystyle\int_0^1 \frac{dt}{1+tz}$.

(a) Show by using Morera's Theorem that f is analytic on the open unit disc $\mathbb{D}$.

(b) Find a power series expansion for $f(z)$ valid on $\mathbb{D}$.

9. Let $P(z)$ and $Q(z)$ be polynomials with degree $Q \geq$ degree $P + 2$. Prove that

$$\sum_{z_i} \operatorname{Res}_{z=z_i} \frac{P(z)}{Q(z)} = 0$$

where the sum is over all poles z_i in $\mathbb{C}$ of the rational function $\dfrac{P}{Q}$.

10. Suppose f is analytic on the punctured unit disc $\mathbb{D} - \{0\}$ and the real part of f is positive there. Prove that f has a removable singularity at 0.