UVM - Analysis Qualifying Exam Topics

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The following is a list of topics found for the Analysis Qualifying Exam at University of Vermont. The topics covered here can be found in Rudin's *Principle's of Mathematical Analysis* and *Real and Complex Analysis*.

Properties of $\mathbf{R}$ and $\mathbf{R}^n$ order axioms; least upper bound property; convergence of bounded monotone sequences; Cauchy sequences; completeness; sup and inf; inner product and distance in $\mathbf{R}^n$; Cauchy–Schwartz inequality and triangle inequality in $\mathbf{R}^n$. Sequences; lim sup and lim inf; series; Cauchy, bounded. Definition of convergent series. Absolute convergence. Tests for convergence: integral, comparison, root, ratio. Counterexamples for root and ratio (inconclusive cases).

Topology of $\mathbf{R}^n$ interior and boundary points; accumulation points; neighborhoods; open and closed sets; connected and path connected sets; compact sets; Bolzano–Weierstrass and Heine–Borel Theorems; metric spaces and topological spaces.

Metric Space Topology open, closed, compact, bounded, etc. Sequential compactness.

Continuity limit of a function at a point; limit, open set and sequential definitions of continuity; continuity in metric spaces; continuity on $\mathbf{R}^n$, especially $\mathbf{R}^1$; continuity of polynomials, rational functions, trig functions; continuous images of compact and connected sets; Intermediate Value Theorem; uniform continuity;

Differentiation in $\mathbf{R}^n$ partial derivatives; Jacobian; chain rule; gradient; directional derivative; Taylor's Theorem with remainder in many variables; maxima and minima, Implicit and Inverse Function Theorems; Lagrange multipliers; Mean Value Theorem; Intermediate Value Theorem for derivatives.

Riemann Integration partitions; upper and lower sums; definition of the integral (upper and lower sums) and Riemann's Condition; integrability of continuous functions (e.g. sketch: f continuous on $[a, b]$ implies $\int_a^b f dx$ exists); measure zero sets; Lebesgue's Theorem; Fundamental Theorem of Calculus; improper integrals; multiple integrals; change of variables; monotone functions and functions of bounded variation; Riemann-Stieltjes integrals.

Sequences and Series of Functions sequences and series of functions; pointwise and uniform convergence; continuity, differentiability and integrability of limits of functions; power series, radius of convergence, convergence tests, Weierstrass M-Test; differentiation and integration of series; spaces of continuous functions, Stone-Weierstrass Theorem, totally bounded sets and compactness in function spaces.

Uniform convergence. Uniform convergence + continuity implies limit is continuous: know how to prove it! Know counterexample for non-uniformly convergent sequence. Weierstrass M-test. Radii of convergence for real power series.

Measure Theory rings, fields, σ-algebras; finite and countably additive measures; definition of measurability; outer measures; Carathéodory extension of a measure; Lebesgue and Lebesgue–Stieltjes measures; distribution functions;

Lebesgue Integration measurable functions; step and simple functions; integrable functions; Definition of integral of non-negative simple functions and of non-negatives non-negative measurable functions; Fatou's Lemma; Dominated and Monotone Convergence Theorems; Riemann integration.

Applications normed linear spaces, Hilbert spaces; L^p-spaces, Hölder and Minkowski inequalities, completeness; dual of L^p, Riesz Representation Theorem; signed measures; Monotone convergence; Fatou's Lemma; Dominated Convergence; differentiation of monotone functions; absolute continuity; Ergorov's Theorem; Radon–Nikodým Theorem.

Density Theorems approximability by continuous functions, step functoins, simple functions, of functions in $L^1(\mathbf{R})$; Riemann-Lebesgue Lemma; Borel-Cantelli; Fubini and Tonelli.

Basic Complex Variables Arithmetic of $\mathbf{C}$, absolute value, polar representation of complex numbers; definition of analytic, Cauchy–Riemann equations (show that if $f'(z)$ exists then $f = u + iv$ satisfies the Cauchy-Riemann equations), and relation to differentiation in $\mathbf{R}^2$; definition and basic properties of trig and exponential functions.

Complex Integration and Power Series paths and line integrals; Goursat's Theorem (sketch proof of; indicate how it leads to Cauchy's Theorem and the Cauchy Integral Formula on the unit disc), Cauchy's Theorem on discs; homotopy and Cauchy's Theorem on simply connected domains; winding number; Cauchy Integral Formula (and how it leads to power series); power series expansions (if a power series centered at a converges at z it converges at all z' which are closer to a than z); Cauchy's estimates for terms of a power series; zeros and poles of analytic functions and their isolation; Open Mapping, Maximum Modulus, Morera, and Liouville Theorems, Fundamental Theorem of Algebra; Laurent expansions; logarithm function.

Zeroes and Singularities Isolated zeroes; orders of zeros and definition and significance; removable vs poles vs essential singularities (know what they mean and relate them to Laurent Series); analytic bounded functions on a punctured disc; Casorati–Weierstrass Theorem; residues; computing residues for simple and multiple poles; computing integrals via residues; meromorphic functions; analytic continuation.

Conformal Mappings and Normal Families compact-open topology, definition of normal families; Montel's Theorem; conformal mapping, Riemann Mapping Theorem and Picard's ("big") Theorem; Schwarz Lemma, analytic automorphisms of the unit disc; simple conformal mappings (e.g. of the upper half plane, sectors, infinite strips, half-disk ONTO the unit disc; fractional linear transformations).

Harmonic Functions definition; existence of harmonic conjugate; mean value property; maximum and minimum principles; Dirichlet Problem on the unit disc.