## ALGEBRA PH.D. QUALIFYING EXAM

September 27, 2008

*A passing paper consists of four problems solved completely plus significant progress on two other problems; moreover, the set of problems solved completely must include one from each of Sections A, B and C.*

### Section A.

*In this section you may quote without proof basic theorems and classifications from group theory and group actions as long as you state clearly what facts you are using.*

**1.** Let $p$ and $q$ be distinct primes and let $G$ be a group of order $p^3q$.

**(a)** Show that if $p > q$ then a Sylow $p$-subgroup of $G$ is normal in $G$.

**(b)** Assume $G$ has more than one Sylow $p$-subgroup. Show that if the intersection of any pair of distinct Sylow $p$-subgroups is the identity, then $G$ has a normal Sylow $q$-subgroup.

**(c)** Assume the Sylow $p$-subgroups of $G$ are abelian. Show that $G$ is not a simple group. (Do not quote Burnside's $p^aq^b$-Theorem.)

**2.** Let $G$ be a finite group acting transitively (on the left) on a nonempty set $\Omega$. For $\omega \in \Omega$ let $G_\omega$ be the usual stabilizer of the point $\omega$:

$$G_\omega = \{g \in G \mid g\omega = \omega\}$$

where $g\omega$ denotes the action of the group element $g$ on the point $\omega$.

**(a)** Prove that $hG_\omega h^{-1} = G_{h\omega}$, for every $h \in G$.

**(b)** Assume $G$ is *abelian*. Let $N$ be the kernel of the transitive action. Prove that $N = G_\omega$ for every $\omega \in \Omega$, and deduce that $|G : N| = |\Omega|$.

**3.** Let $N$ be a normal subgroup of the group $G$ and for each $g \in G$ let $\phi_g$ denote conjugation by $g$ acting on $N$, i.e.,

$$\phi_g(x) = gxg^{-1} \qquad \text{for all } x \in N.$$

**(a)** Prove that $\phi_g$ is an automorphism of $N$ for each $g \in G$.

**(b)** Prove that the map $\Phi : g \mapsto \phi_g$ is a homomorphism from $G$ into $\mathrm{Aut}(N)$, where $\mathrm{Aut}(N)$ is the automorphism group of $N$.

**(c)** Prove that $\ker \Phi = C_G(N)$ and deduce that $G/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$.

### Section B.

**4.** Let $X$ be any nonempty set and let $R$ be the (commutative) ring of all integer-valued functions on $X$ under the usual pointwise operations of addition and multiplication of functions:
$R = \{f \mid f : X \longrightarrow \mathbb{Z}\}$. For each $a \in X$ define $M_a = \{f \in R \mid f(a) = 0\}$.

**(a)** Prove that $M_a$ is a prime ideal in $R$.

**(b)** Prove that $M_a$ is not a maximal ideal in $R$.

**(c)** Find all units in $R$.

**(d)** Find all zero divisors in $R$.

**5.** Let $F$ and $K$ be finite fields with $F \subseteq K$. Let $F[x]$ and $K[x]$ denote the respective polynomial rings in the variable $x$, so $F[x]$ is a subring of $K[x]$.

**(a)** Prove that if $M$ is any maximal ideal in $K[x]$, then $M \cap F[x]$ is a maximal ideal in $F[x]$.

**(b)** Give an explicit example of commutative rings $A \subseteq B$ and a maximal ideal $I$ of $B$ such that $I \cap A$ is not a maximal ideal of $A$.

**6.** Let $R$ be a Principal Ideal Domain, let $p$ and $q$ be distinct primes in $R$, and let $a = p^\alpha q^\beta$ for some $\alpha, \beta \in \mathbb{Z}^+$. Let $M$ be any $R$-module annihilated by $(a)$. Prove that

$$M \cong M_p \oplus M_q$$

where $M_p$ is the submodule of $M$ annihilated by $(p^\alpha)$ and $M_q$ is the submodule of $M$ annihilated by $(q^\beta)$.

**7. (a)** How many similarity classes of $8 \times 8$ matrices $A$ with rational number entries are there that satisfy $A^8 = A$? (Explain briefly; you need not explicitly list all classes.)

**(b)** How many similarity classes of $3 \times 3$ matrices $A$ with entries from the field $\mathbb{Z}/7\mathbb{Z}$ are there that satisfy $A^8 = A$? (Explain briefly; you need not explicitly list all classes.)

## Section C.

**8.** Let $E$ be the splitting field in $\mathbb{C}$ of the polynomial $p(x) = x^6 + 3x^3 - 10$ over $\mathbb{Q}$, and let $\alpha$ be any root of $p(x)$ in $E$.

**(a)** Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

**(b)** Describe the roots of $p(x)$ in terms of radicals involving rational numbers and roots of unity.

**(c)** Find $[E : \mathbb{Q}]$. (Justify)

**(d)** Prove that $E$ contains a *unique* subfield $F$ with $[F : \mathbb{Q}] = 2$.

**9.** Let $p$ be a prime, let $F$ be a field of characteristic 0, let $E$ be the splitting field over $F$ of an irreducible polynomial of degree $p$, and let $G = Gal(E/F)$.

**(a)** Explain why $[E : F] = pm$ for some integer $m$ with $(p, m) = 1$.

**(b)** Prove that if $G$ has a normal subgroup of order $m$, then $[E : F] = p$ (i.e., $m = 1$).

**(c)** Assume $p = 5$ and $E$ is *not* solvable by radicals over $F$. Show that there are exactly 6 fields $K$ with $F \subseteq K \subseteq E$ and $[E : K] = 5$.
(You may quote without proof basic facts about groups of small order.)

**10.** Let $p$ be a prime, let $\mathbb{F}_p$ be the field of order $p$, and let $f(x)$ be a nonconstant polynomial in $\mathbb{F}_p[x]$. Assume $f$ factors as

$$f(x) = q_1(x)^{\alpha_1} q_2(x)^{\alpha_2} \cdots q_r(x)^{\alpha_r} \tag{10.1}$$

for some distinct irreducible polynomials $q_1, \ldots, q_r$ in $\mathbb{F}_p[x]$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$. Let $E$ be a splitting field of $f$ over $\mathbb{F}_p$.

**(a)** Give an expression for $[E : \mathbb{F}_p]$ in terms of the $q_i$ in (10.1).
[Hint: Your answer should only involve the degrees of the $q_i$'s, and not depend on the $\alpha_i$'s.]

**(b)** Fix a natural number $N$ and assume $q_1, \ldots, q_r$ are *all* the distinct irreducible polynomials of degree $\leq N$ in $\mathbb{F}_p[x]$. Find an expression for $[E : \mathbb{F}_p]$ in terms of $N$, where $f$ is as in (10.1).