ALGEBRA PH.D. QUALIFYING EXAM

May 20, 2008

A passing paper consists of four problems solved completely plus significant progress on two other problems; moreover, the set of problems solved completely must include one from each of Sections A, B and C.

Section A.

In this section you may quote without proof basic theorems and classifications from group theory and group actions as long as you state clearly what facts you are using.

1. Let G be a *solvable* group of order $168 = 2^3 \cdot 3 \cdot 7$. The aim of this exercise is to show that G has a normal Sylow p-subgroup for some prime p. Let M be a minimal normal subgroup of G.

- **(a)** Show that if M is not a Sylow p-subgroup for any prime p, then $|M| = 2$ or 4.
[You may quote without proof any results you need about minimal normal subgroups of solvable groups.]
- **(b)** Assume $|M| = 2$ or 4, and let $\overline{G} = G/M$. Prove that $\overline{G}$ has a normal Sylow 7-subgroup.
- **(c)** Under the same assumptions and notation as (b), let H be the complete preimage in G of the normal Sylow 7-subgroup of $\overline{G}$. Prove that H has a normal Sylow 7-subgroup, P, and deduce that P is normal in G.

2. Let G be a finite group acting transitively (on the left) on a nonempty set Ω. Let $N \trianglelefteq G$, and let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ be the orbits of N acting on Ω. For any $g \in G$ let $g\mathcal{O}_i = \{g\alpha \mid \alpha \in \mathcal{O}_i\}$.

- **(a)** Prove that $g\mathcal{O}_i$ is an orbit of N, for any $i \in \{1, 2, \ldots, r\}$, i.e., $g\mathcal{O}_i = \mathcal{O}_j$ for some j.
- **(b)** Explain why G permutes $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$ *transitively* (acting as in part (a)).
- **(c)** Deduce from (b) that $r = |G : NG_\alpha|$, where G_α is the subgroup of G stabilizing the point $\alpha \in \mathcal{O}_1$.

3. Let G be a finite group, let p be a prime and let $P \in Syl_p(G)$. Assume P is abelian.

- **(a)** Prove that two elements of P are conjugate in G if and only if they are conjugate in $N_G(P)$.
- **(b)** Prove that $P \cap gPg^{-1} = 1$ for every $g \in G - N_G(P)$ if and only if $P \trianglelefteq C_G(x)$ for every nonidentity element $x \in P$.

Section B.

4. Let $F_1, F_2, \ldots, F_n$ be fields.

- **(a)** Explicitly describe all the ideals in the direct product ring $F_1 \times F_2 \times \cdots \times F_n$.
(Explain briefly why your list is complete.)
- **(b)** Which of the ideals in (a) are prime? (Justify.)
- **(c)** Which of the ideals in (a) are maximal? (Justify.)

5. Let R be an integral domain and assume R contains a subring F that is a field (R and F have the same 1). Prove that if R is finite dimensional as a vector space over F then R is a field.

6. Let R be a commutative ring with 1 and let A, B and C be left R-modules. Prove that $\mathrm{Hom}_R(A, B \oplus C) \cong \mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A, C)$, where this is an isomorphism of R-modules.

7. (a) How many similarity classes of 8×8 matrices A with rational number entries are there that satisfy $A^8 = I$ but $A^n \neq I$ for every $n \in \{1, \ldots, 7\}$? (Justify.)

(b) Answer the same question as in part (a) but with the field of rational numbers replaced by the field $\mathbb{F}_2$ with 2 elements. (Justify.)

Section C.

8. Let E be the splitting field in $\mathbb{C}$ of the polynomial $p(x) = x^6 + 3x^3 + 3$ over $\mathbb{Q}$, and let α be any root of $p(x)$ in E.

(a) Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.

(b) Show that $\alpha^3 + 1 = \omega$ is a primitive cube root of unity. Describe the roots of $p(x)$ in terms of radicals involving rational numbers and ω.

(c) Assume $E \neq \mathbb{Q}(\alpha)$, and prove $[E : \mathbb{Q}] = 18$.
[Hint: Show first that $\mathbb{Q}(\beta)$ is a Galois extension of $\mathbb{Q}(\omega)$ of degree 3, for every root β of $p(x)$.]

(d) Again assume $E \neq \mathbb{Q}(\alpha)$, and prove that E contains a *unique* subfield F with $[F : \mathbb{Q}] = 2$.

9. Let K/F be an extension of odd degree, where F is any field of characteristic 0.

(a) Let $a \in F$ and assume the polynomial $x^2 - a$ is irreducible over F. Prove that $x^2 - a$ is also irreducible over K.

(b) Assume further that K is Galois over F. Let $\alpha \in K$ and let E be the Galois closure of $K(\sqrt{\alpha})$ over F. Prove that $[E : F] = 2^r [K : F]$ for some $r \geq 0$.

10. Let p be a prime, let $F = \mathbb{F}_p$ be the field of order p, and let $\overline{F}$ be an algebraic closure of F. Let n be a positive integer relatively prime to p and let F_n be the splitting field of the polynomial $f_n(X)$ in $\overline{F}$, where

$$f_n(X) = X^n - 1.$$

(a) Explain briefly why $[F_n : F]$ is equal to the order of p in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$.
[Quote without proof basic facts you need about finite fields.]

(b) If n and m are relatively prime and neither is divisible by p, is $F_{nm} = F_n F_m$?
(Justify briefly.)