ALGEBRA PH.D. QUALIFYING EXAM

January 17, 2019

Three hours

A passing paper consists of five problems solved completely, or four solved completely plus significant progress on two other problems; in both cases the set of problems solved completely must include one from each of Sections A, B and C.

Section A.

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Let G be a group of order 2457 (note that $2457 = 3^3 \cdot 7 \cdot 13$).

(a) Compute the number, n_p, of Sylow p-subgroups permitted by Sylow's Theorem for each of $p = 7$ and 13 (only).

(b) Let P_{13} be a Sylow 13-subgroup of G. Prove that if P_{13} is not normal in G, then $N_G(P_{13})$ has a normal Sylow 7-subgroup.

(c) Deduce from (b) and (a) that G has a normal Sylow p-subgroup for either $p = 13$ or $p = 7$.

2. Let p be a prime and let P be a p-group acting on a nonempty finite set A with $(|\,A\,|, p) = 1$.

(a) Prove that there is some $a \in A$ that is fixed by every element of P.

(b) Suppose P is a p-subgroup of a finite group G and H is a normal subgroup of G with $(|\,H\,|, p) = 1$. Deduce from (a) that for every prime q dividing $|\,H\,|$ there is a Sylow q-subgroup of H that is normalized by P.

3. Let G be a group containing nonabelian simple subgroups H_i such that

$$H_1 \leq H_2 \leq H_3 \leq \cdots \qquad \text{and} \qquad \bigcup_{n=1}^{\infty} H_n = G. \tag{3}$$

(a) Prove that G is simple.

(b) Prove that if $H_n \neq H_{n+1}$ for all n, then G is not finitely generated.

Section B.

4. Let R be the ring of all *continuous* real valued functions on the closed interval [0,1]. For each $a \in [0, 1]$ let $M_a = \{f \in R \mid f(a) = 0\}$.

(a) Find all units in R.

(b) Give an explicit example of a nonzero zero divisor in R.

(c) Prove that M_a is a maximal ideal in R.

(d) Prove that there is a countable subset $\{a_1, a_2, a_3, \dots\}$ of [0,1] such that $\bigcap_{i=1}^{\infty} M_{a_i} = 0$.

5. Let R be a Principal Ideal Domain with field of fractions F and assume $R \neq F$. As usual we may view F as a module over its subring R.

(a) Prove that every finitely generated R-submodule of F is a cyclic R-module.

(b) Deduce from (a) that F cannot be a finitely generated R-module.

(You may quote results about modules over PIDs.)

6. Let $\mathbb{F}_q$ be the finite field with q elements. Find the number of similarity classes of 5×5 matrices A over F_q that satisfy $A^q = I$, where I is the identity matrix. (Justify your answer. You do not need to exhibit representatives of the classes.)

Section C.

7. Let $f(x) = x^6 - 6x^3 + 1$ and let α, β be the two real roots of $f(x)$ with $\alpha > \beta$. You may assume $f(x)$ is irreducible in $\mathbb{Q}[x]$. Let K be the splitting field of $f(x)$ in $\mathbb{C}$.

(a) Exhibit all six roots of $f(x)$ in terms of radicals involving only integers and powers of ω, where ω is a primitive cube root of unity.

(b) Prove that $K = \mathbb{Q}(\alpha, \omega)$ and deduce that $[K : \mathbb{Q}] = 12$. [Hint: What is $\alpha\beta$?]

(c) Prove that $G = \mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup N such that G/N is the Klein group of order four.

8. Let n be a given positive integer and let E_{2^n} be the elementary abelian group of order 2^n (the direct product of n copies of the cyclic group of order 2). Show that there is some positive integer N such that the cyclotomic field $\mathbb{Q}(\zeta_N)$ contains a subfield F that is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(F/\mathbb{Q}) \cong E_{2^n}$, where ζ_N is a primitive N^{th} root of 1 in $\mathbb{C}$.

9. Let p be a prime and let $q = p^n$ for some $n \in \mathbb{Z}^+$.

(a) What is the degree of the extension $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$? Describe how the Frobenius automorphism, σ, for this extension acts on the elements of $\mathbb{F}_{q^2}$.

(b) Define the norm map

$$N : \mathbb{F}_{q^2}^{\times} \longrightarrow \mathbb{F}_q^{\times} \qquad \text{by} \qquad N(a) = a\sigma(a).$$

Prove that this norm map is surjective. [Hint: Note that N is a homomorphism of multiplicative groups. Use (a) and facts about finite fields to find the order of its kernel.]