There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let $\mathbb{D}$ denote the open disc of radius 1 centered at the origin.

1. Let $f(z) = z|z|^2$.

   (a) Find all points in the complex plane where $f$ satisfies the Cauchy-Riemann equations.

   (b) Does $f$ have a complex derivative at the points you found in (a)? (Justify briefly.)

   Solution: (a): One finds $u = x^3 + xy^2$, $v = x^2y + y^3$. The Cauchy-Riemann Equations require $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This gives $3x^2 + y^2 = x^2 + 3y^2$ and $2xy = -2xy$. The latter implies $x = 0$ or $y = 0$, and then the former yields $x = 0$ and $y = 0$. The only point satisfying C-R is $z = 0$.

   (b): Since both partials are continuous at $z = 0$ and C-R are satisfied, $f$ does have a complex derivative there, namely $f'(0) = 0$. This can also be obtained directly as the limit of a difference quotient.

2. Find all complex numbers $z$ such that $\tan z = i - 1$.

   Solution: We use $\tan(z) = \sin(z)/\cos(z) = -i(e^{2iz} - 1)/(e^{2iz} + 1)$. Substituting $w = e^{2iz}$, we obtain an easily-solved linear equation for $w$, yielding $w = (1 + 2i)/5$. Then solving $e^{2iz} = (1 + 2i)/5$, gives

   $$2iz = -\frac{1}{2} \log(5) + i(\arg((1 + 2i)/5) + 2\pi n), \quad n \in \mathbb{Z}$$

   so $z = i \log(5)/4 + \arg(1 + 2i)/2 + \pi n$, for any integer $n$.

3. Let $f(z) = z$. In each of parts (a) and (b) compute $\int_\gamma f(z)\,dz$, where $\gamma$ is the specified path whose initial point is $-1$ and terminal point is $i$.

   (a) $\gamma$ is the path along the coordinate axes: from $-1$ to 0 and then from 0 to $i$.

   (b) $\gamma$ is the quarter of the unit circle lying in the second quadrant, oriented clockwise.

   (c) Could there be a function $g$ that is analytic on some simply connected open set $U$ containing both the paths in (a) and (b) such that $g = f$ on both paths? (Explain briefly.)

   Solution: (a): Parameterizing the paths leads to $\int_{-1}^{0} t\,dt + \int_{0}^{1} (-it)i\,dt = -1/2 + 1/2 = 0$
(b): Using \( z = -e^{-it}, 0 \leq t \leq \pi/2 \) leads to \( \int_{0}^{\pi/2} -e^{it}i e^{-it} \, dt = -i\pi/2. \)

(c): No. Integrals of any analytic function \( g \) over paths contained in a simply connected domain that have a common initial point and common terminal point result in the same value, contrary to what happened in (a) and (b).

4. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point \( z = 3 \) for the following function (explain briefly how the inner and outer radii of the annulus are determined):

\[
f(z) = \frac{z}{z^2 - 4} + \frac{12}{(z - 4)^2}.
\]

**Solution:** We can write \( f(z) = \frac{1}{z} - \frac{1}{z^2} + \frac{3/4}{(1 - (z/4)^2)^2} \). Using the geometric series and its derivative, we find

\[
f(z) = \left( \frac{1}{z} + \frac{4}{z^3} + \frac{4^2}{z^5} + \cdots \right) + \left( \frac{3}{4} + \frac{3}{4} (\frac{z}{4}) + \frac{3}{4} (\frac{z}{4})^2 + \cdots \right)
\]

The radii are determined as the min and max radii such that the open annulus they determine contains \( z = 3 \) and excludes all (non-removable) singularities of \( f(z) \). In this case the radii are 2 and 4.

5. Use the calculus of residues to evaluate the improper integral

\[
\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 9} \, dx.
\]

**[Draw your path of integration; but you may quote without proof any standard estimates for integrals along portions of your path, making sure to mention what growth conditions are required.]**

**Solution:** Let \( f(z) = z e^{iz}/(z^2 + 9) \) so that the specified integral is the imaginary part of \( \int_{-\infty}^{\infty} f(z) \, dz \). This is a standard Fourier transform integral of the type \( \int_{-\infty}^{\infty} g(z) e^{iaz} \, dz \), where \( a = 1 \) and \( g(z) = z/(z^2 + 9) \). One only needs the growth condition

\[
|g(z)| \leq K/|z| \quad \text{for some } K \in \mathbb{R}^+
\]

for all sufficiently large \(|z|\) in order to integrate \( f \) over a rectangular path whose lower horizontal base is \([-A, B]\) on the real axis. The integrals along the remaining three sides go to zero as their lengths tend to infinity (see, for example, Lang’s *Complex Analysis*, Section VI.2). Then \( \int_{-\infty}^{\infty} f(z) \, dz = 2\pi i (\text{sum of the residues of } f \text{ in the UHP}) \). The only poles of \( f \) are at \( \pm 3i \) with only \( 3i \) in the UHP. The residue at this simple pole is \( e^{-3}/2 \). Thus the value of the integral is the imaginary part of \( 2\pi i e^{-3}/2 \) (which is pure imaginary), i.e., the integral is \( \pi/e^3 \).
6. Suppose $f$ is entire and there is a positive real number $M$ and a polynomial $p$ such that $|f(z)| \leq |p(z)|$ for all $z$ with $|z| > M$. Prove that $f$ is a polynomial.

**Solution:** If $p$ is a polynomial of degree $d$, then there are positive real constants $A$ and $B$ such that $|p(z)| \leq A + B|z|^d$ for all $z$ with $|z| > M$. By a standard Cauchy estimate, the $n^{th}$ derivative of $f$ vanishes at 0 for all $n > d$. Thus the Taylor series for $f$ expanded about the origin is a polynomial.

7. Let $f$ and $g$ be analytic on the closed unit disc $\overline{D}$, and assume both $f$ and $g$ have no zeros in $D$. Prove that if $|f(z)| = |g(z)|$ for all $z$ with $|z| = 1$, then $f(z) = kg(z)$ in $D$ for some constant $k$ of modulus 1.

**Solution:** It is immediate from the hypotheses that both $f/g$ and $g/f$ are analytic on $D$. By the Maximum Modulus Principle both $|f/g|$ and $|g/f|$ are bounded by 1 on $D$, hence $|f/g| = |g/f| = 1$ on $D$. Since $f/g$ then has a maximum modulus in the open connected set $D$, $f/g$ must be constant on $D$, hence $f/g = k$ for some $k \in \mathbb{C}$ of modulus 1.

8. (a) Exhibit a function $f$ such that at each positive integer $n$, $f$ has a pole of order $n$, and $f$ is analytic and nonzero at every other complex number. (Briefly justify your answer.)

(b) Let $f$ be any function that satisfies the conditions of part (a). For each positive integer $N$ find $\int_{C_N} \frac{f'(z)}{f(z)} \, dz$, where $C_N$ is the circle of radius $N + \frac{1}{2}$ centered at the origin.

**Solution:** First let $z_n$ be the $n^{th}$ term in the sequence $1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$. Note that
\[
\sum_{n=1}^{\infty} \frac{1}{|z_n|^3} = \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]
Thus by standard results on Weierstrass products the function
\[
g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp(z/z_n + (z/z_n)^2/2)
\]
is entire and has zeros only at the positive integers, with the zero at $n$ of order $n$, for all positive integers $n$. Thus $f = 1/g$ satisfies the conditions of (a).

(b): For any meromorphic function $f$, the poles of $f'/f$ occur exactly at the points $z_0$ that are zeros or poles of $f$; and $f'/f$ then has a simple pole at $z_0$ with residue equal to the order of zero or pole of $f$ at $z_0$. For $f$ as in (a), the only poles of $f'/f$ are therefore at each positive integer $n$ with residue $-n$. Exactly the first $N$ of these lie inside $C_N$. By the Residue Theorem
\[
\int_{C_N} \frac{f'(z)}{f(z)} \, dz = 2\pi i \sum_{n=1}^{N} -n = -\pi in(n + 1).
\]
9. (a) State Goursat’s Theorem.

(b) Use Goursat’s Theorem to prove that if $f$ is continuous on $\mathbb{C}$ and analytic at every point not on the real axis, then $f$ must be analytic everywhere.

**Solution:** (a): Look it up.

(b): By Goursat’s Theorem it suffices to show that the integral of $f$ around any rectangle, $R$, whose sides are parallel to the axes is 0. If $R$ lies entirely in either the upper or the lower half-plane, this is true by Goursat since $f$ is analytic in these regions. If the vertical sides of $R$ cross the real axis, divide $R$ into two rectangles, $R_1, R_2$, that share a side lying on the real axis: the integral around $R$ is the same as the sum of the integrals around $R_1$ and $R_2$ (the integrals along the real axis sides cancel because they are in reverse directions).

Thus we are reduced considering only the case when $R$ equals $R_1$ or $R_2$, i.e., $R$ has a side on the real axis. By symmetry we may assume $R$ lies in the closed upper half-plane. Let $\epsilon > 0$ be given; we show $|\int_R f \, dz| < \epsilon$. As before, divide $R$ into two rectangles, $R_1, R_2$, sharing a common side: $R_1$ entirely in the upper half-plane, and $R_2$ whose horizontal sides are the intervals $[a, b]$ and $[a+\delta i, b+\delta i]$ for some small $\delta$ to be determined. As before, $\int_{R_1} f \, dz = 0$. We finish the proof by showing $|\int_{R_2} f \, dz| < \epsilon$ for all sufficiently small $\delta$. This is straightforward because by uniform continuity of $f$ on compact sets containing $R_2$ we can choose $\delta$ sufficiently small so that $|f(x) - f(x + \delta i)| < \epsilon/2(b - a)$ for all $x \in [a, b]$. And by choosing $\delta$ sufficiently small, we can arrange that the integral of $f$ over the vertical sides (each of length $\delta$) is $< \epsilon/2$. Since the integrals along the top and bottom sides are in reverse directions, these estimates show $\int_{R_2} |f| \, dz$ can be made less than any given $\epsilon$.

10. Suppose $f$ is entire and for some positive real number $K$

$$|\text{Re } f(z)| \geq |\text{Im } f(z)|, \quad \text{for all } z \text{ with } |z| \geq K.$$ 

Prove that $f$ is constant on $\mathbb{C}$.

**Solution:** Let $g(z) = f(1/z)$ so the singularity of $f$ at infinity is the same as the isolated singularity of $g$ at 0, and the hypotheses for $f$ hold likewise for $g$ in some small disc about 0. Note that if $f$ has a removable singularity at infinity, then it is entire and bounded, hence constant by Liouville. One must therefore show $f$ does not have a pole or essential singularity at infinity; equivalently, $g$ does not have a pole or essential singularity at 0. By hypothesis and Casorati-Weierstrass $g$ does not have an essential singularity at 0 since the values of $g$ would get arbitrarily close to $i$ in any neighborhood of 0, thereby violating the hypothesis. Suppose $g$ has a pole of order $n \geq 1$ at 0. In this case, the Laurent expansion of $g$ about 0 gives

$$(10.1) \quad g(z) = \frac{a}{z^n} (1 + h(z)) \quad \text{for some } a \in \mathbb{C} - \{0\}$$

in some neighborhood of zero where $h$ is analytic with $h(0) = 0$. For $M$ a positive real number let $z_M$ be any $n^{th}$ root of $\frac{a}{M}$. Now choose $M > 4$ and large enough so that $z_M$ is in the neighborhood of convergence of $(10.1)$, and $|1/z_M| \geq K$, and $|h(z_M)| < 1/4$. Then $g(z_M) = Mi(1 + h(z_M))$ is seen to violate the condition that $|\text{Re } g(z_M)| \geq |\text{Im } g(z_M)|$, a contradiction.

One may also solve this question by invoking the Open Mapping Theorem.