COMPLEX VARIABLES PH.D. QUALIFYING EXAM

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There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let \( \mathbb{D} \) denote the open disc of radius 1 centered at the origin.

1. Let \( f \) be holomorphic on a connected open set \( U \). Prove that if \( f(z)^2 = \overline{f(z)} \) for all \( z \in U \) then \( f \) is constant on \( U \). Find all possible values for \( f \).

Solution: Write \( f(z) = u(z) + iv(z) \) for real valued functions \( u \) and \( v \) on \( U \). Then

\[
  f^2 = (u^2 - v^2) + i(2uv) = u - iv = \overline{f}.
\]

Equating imaginary parts gives \( u = -1/2 \) whenever \( v \neq 0 \). When \( v = 0 \), equating real parts gives \( u = 0 \) or \( 1 \). Thus the continuous function \( u \) takes at most three values on the connected set \( U \), hence must be constant there. By Cauchy–Riemann, \( v \) is likewise constant on \( U \). If \( v \) is identically zero on \( U \), then \( f(z) = 0 \) or \( 1 \) on \( U \). If \( v \neq 0 \) on \( U \) then \( u = -1/2 \) and so \( v^2 = 3/4 \). In this case \( f(z) = (-1 \pm i\sqrt{3})/2 \), which are the two primitive cube roots of unity.

Alternatively, \( f^3 = |f|^2 \) shows that \( f^3 \) is non-negative real and of absolute value 0 or 1. Thus the only values \( f \) may take are 0 and cube roots of unity. Since \( f \) is continuous on a connected set, its image is connected and so must be constant.

2. Let \( \gamma \) be the circle of radius 5 centered at 0. Evaluate with brief justification the integrals:

(a) \( \int_{\gamma} \frac{z}{z - 1} \, dz \),  
(b) \( \int_{\gamma} e^{1/z} \, dz \).

Solution: (a): On \( \gamma \) we have \( z \bar{z} = 25 \) so \( \bar{z} = 25/z \). Thus

\[
  \int_{\gamma} \frac{z}{z - 1} \, dz = \int_{\gamma} \frac{25}{z(z - 1)} \, dz = \int_{\gamma} \frac{25}{z - 1} - \frac{25}{z} \, dz = 2\pi i(25 - 25) = 0
\]

by the Cauchy Integral Formula (or otherwise).

(b): On and inside the disc bounded by \( \gamma \) the function \( e^{1/z} \) has a Laurent series expansion (plug \( 1/z \) into the exponential power series) which converges uniformly on any region \( |z| > \epsilon \), for \( \epsilon > 0 \). Moreover \( e^{1/z} = 1/z + h(z) \) where \( h \) has a primitive on that region. Thus

\[
  \int_{\gamma} e^{1/z} \, dz = \int_{\gamma} \frac{1}{z} \, dz + \int_{\gamma} h(z) \, dz = 2\pi i + 0.
\]

Alternatively, some versions of the Residue Theorem cover this situation and it is just a matter of finding the residue at \( z = 0 \).
3. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point \( z = 2 \) for the following function (explain briefly how the inner and outer radii of the annulus are determined):

\[
f(z) = \frac{z}{1-z^2} + \frac{6}{(z-4)^2}.
\]

**Solution:** As usual, write

\[
f(z) = -\frac{z}{z^2} \frac{1}{1-(1/z^2)} + \frac{1}{16} \frac{6}{(1-(z/4))^2}
\]

Since \( 1/(1-u)^2 = \frac{d}{du} 1/(1-u) \) we obtain from familiar geometric series expansions:

\[
f(z) = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^{2n}} + \frac{3}{8} \sum_{n=1}^{\infty} n(z/4)^{n-1}.
\]

The first series converges for \( |z| > 1 \) and the second for \( |z| < 4 \).

4. Use the calculus of residues to evaluate the improper integral

\[
\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1} \, dx.
\]

**Solution:** For \( R > 0 \) let \( C_R \) be the semicircle in the upper half plane with diameter the interval \([-R, R]\), let \( \gamma_R \) be the closed curve consisting of \( C_R \) and \([-R, R]\) (oriented counterclockwise), and let \( f(z) = e^{2iz}/(z^2 + 1) \). Since \( \cos(2x) = \text{Re}(e^{2ix}) \), the integral may be found by taking the real part of \( \int_{\gamma_R} f(z) \, dz \) as \( R \to \infty \). This is because, by familiar estimates, the integral of \( f \) along \( C_R \) tends to 0 as \( R \to \infty \). For all \( R > 1 \), \( f(z) \) has one pole inside \( \gamma_R \): a simple pole at \( z = i \) with residue \( e^{-2}/2i \). Thus the specified integral is \( 2\pi i (e^{-2}/2i) = \pi/e^2 \) (which is already a real number).

5. Prove that if \( f \) is entire and there are positive real numbers \( A, B \) and \( k \) such that \( |f(z)| \leq A + B|z|^k \) for all \( z \in \mathbb{C} \), then \( f \) is a polynomial.

**Solution:** By a standard Cauchy estimate, the \( N \)th derivative of \( f \) vanishes at 0 for all \( N > k \). Thus the Taylor series for \( f \) expanded about the origin is a polynomial.

6. Let \( f \) be analytic on the closed unit disc \( \overline{\mathbb{D}} \), and assume \( |f(z)| < 1 \) on its boundary. Prove that there is one and only one point \( z_0 \in \mathbb{D} \) such that \( f(z_0) = z_0 \).

**Solution:** Apply Rouché’s Theorem to the functions \( f(z) - z \) and \( -z \) to obtain this.
7. (a) Exhibit an entire function, $P(z)$, that has simple zeros at the numbers $\sqrt{n}$ for each positive integer $n$, and no other zeros.

(b) For the function $P$ you gave in part (a), describe $P'/P$ as an infinite series (not necessarily a Taylor series however).

Solution: Since $\sum_{n=1}^{\infty} (1/\sqrt{n})^3$ converges, the following Weierstrass product is entire:

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\sqrt{n}} \right) \exp \left( \frac{z}{\sqrt{n}} + \frac{1}{2} \left( \frac{z}{\sqrt{n}} \right)^2 \right)$$

Its logarithmic derivative is

$$\frac{P'}{P}(z) = \sum_{n=1}^{\infty} \frac{z^2}{n(z - \sqrt{n})}.$$ 

Alternatively, one may know that $1/\Gamma(z)$ is entire with simple zeroes at the non-positive integers. Thus $1/\Gamma(1 - z^2)$ provides an example of the desired type. One then needs to know the product formula for $\Gamma(z)$, which leads to a similar infinite series.

8. Define $f(z) = \int_0^1 \frac{dt}{1 + tz}$.

(a) Show by using Morera’s Theorem that $f$ is analytic on the open unit disc $\mathbb{D}$.

(b) Find a power series expansion for $f(z)$ valid on $\mathbb{D}$.

Solution: (a): Let $\gamma$ be a closed curve in $\mathbb{D}$ (it is sufficient to consider $\gamma$ just a triangle or rectangle). Since $1/(1 + tz)$ is continuous on the compact set $\gamma \times [0,1]$ it is permissible to interchange order of integration:

$$\int_\gamma f(z) \, dz = \int_\gamma \int_0^1 \frac{dt}{1 + tz} \, dz = \int_0^1 \int_\gamma \frac{1}{1 + tz} \, dz \, dt.$$ 

For every $t > 0$ the integrand has a primitive $\frac{1}{t} \log_\gamma(1 + tz)$ in the variable $z$ on $\mathbb{D}$, and so each inner integral is zero by Cauchy (we can discard the single point $t = 0$ when computing the outer integral). By Morera $f$ is analytic on $\mathbb{D}$.

(b): By using the familiar power series expansion for $1/(1 + u)$ we obtain

$$f(z) = \int_0^1 \frac{dt}{1 + tz} = \int_0^1 \left( \sum_{n=0}^{\infty} (-1)^n (tz)^n \right) \, dt = \sum_{n=0}^{\infty} \left( \int_0^1 (-1)^n t^n \, dt \right) z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^n.$$
9. Let \( P(z) \) and \( Q(z) \) be polynomials with degree \( Q \geq \) degree \( P + 2 \). Prove that

\[
\sum_{z_i} \text{Res}_{z=z_i} \frac{P(z)}{Q(z)} = 0
\]

where the sum is over all poles \( z_i \) in \( \mathbb{C} \) of the rational function \( \frac{P}{Q} \).

**Solution:** Let \( C_R \) be the circle of radius \( R \) centered at the origin. By cancelling common factors if necessary, we may assume \( P/Q \) is in lowest terms, so its poles occur at the zeros of \( Q \). For any \( R \) greater than the largest modulus of a zero for \( Q \), by the Residue Theorem

\[
2\pi i \sum_{z_i} \text{Res}_{z=z_i} \frac{P(z)}{Q(z)} = \int_{C_R} \frac{P(z)}{Q(z)} \, dz.
\]

The length of \( C_R \) is \( 2\pi R \); and it follows easily from the hypothesis that the integrand is asymptotically bounded in modulus by a constant times \( 1/R^2 \). Thus the usual estimates on the size of \( \int_{C_R} P/Q \, dz \) show that this integral tends to zero as \( R \to \infty \). Since the left hand side of (9.1) is constant, it must be 0, as needed.

10. Suppose \( f \) is analytic on the punctured unit disc \( \mathbb{D} - \{0\} \) and the real part of \( f \) is positive there. Prove that \( f \) has a removable singularity at 0.

**Solution:** Consider \( g = (1 - f)/(1 + f) \), which is bounded by 1 on the set where Re \( f > 0 \). The filled-in value of \( g(0) \) is some \( b \) with \( |b| \leq 1 \). By the Maximum Principle (\( f \) isn’t identically infinite, therefore \( |g| < 1 \) for \( z \) near \( 0 \)), \( |b| < 1 \). The filled-in value of \( f(0) \) is \( g^{-1}(b) \), which is a finite complex number.

Alternatively, by Casorati-Weierstrass \( f \) does not have an essential singularity at 0. If \( f \) has a pole at 0, one can write \( f = cz^{-n}(1 + h(z)) \) where \( h \) is analytic and \( h(0) = 0 \). Then it is easy to see that the real part of \( f \) must be negative arbitrarily close to 0.

Or, one can consider \( g(z) = e^{-f(z)} \) at zero (which is bounded and analytic on the punctured disc), taking special care to eliminate the case when \( g(0) = 0 \).