COMPLEX VARIABLES PH.D. QUALIFYING EXAM

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There are ten questions. A passing paper consists of seven problems done completely correctly, or six problems done correctly with substantial progress on two others. Let $\mathbb{D}$ denote the open disc of radius 1 centered at the origin.

1. Let f be holomorphic on a connected open set U. Prove that if $f(z)^2 = \overline{f(z)}$ for all $z \in U$ then f is constant on U. Find all possible values for f.

Solution: Write $f(z) = u(z) + iv(z)$ for real valued functions u and v on U. Then

$$f^2 = (u^2 - v^2) + i(2uv) = u - iv = \overline{f}.$$

Equating imaginary parts gives $u = -1/2$ whenever $v \neq 0$. When $v = 0$, equating real parts gives $u = 0$ or 1. Thus the continuous function u takes at most three values on the connected set U, hence must be constant there. By Cauchy–Riemann, v is likewise constant on U. If v is identically zero on U, then $f(z) = 0$ or 1 on U. If $v \neq 0$ on U then $u = -1/2$ and so $v^2 = 3/4$. In this case $f(z) = (-1 \pm i\sqrt{3})/2$, which are the two primitive cube roots of unity.

Alternatively, $f^3 = |f|^2$ shows that f^3 is non-negative real and of absolute value 0 or 1. Thus the only values f may take are 0 and cube roots of unity. Since f is continuous on a connected set, its image is connected and so must be constant.

2. Let γ be the circle of radius 5 centered at 0. Evaluate with brief justification the integrals:

(a) $\displaystyle\int_\gamma \frac{\overline{z}}{z-1}\, dz,$ (b) $\displaystyle\int_\gamma e^{1/z}\, dz.$

Solution: (a): On γ we have $z\overline{z} = 25$ so $\overline{z} = 25/z$. Thus

$$\int_\gamma \frac{\overline{z}}{z-1}\, dz = \int_\gamma \frac{25}{z(z-1)}\, dz = \int_\gamma \frac{25}{z-1} - \frac{25}{z}\, dz = 2\pi i(25-25) = 0$$

by the Cauchy Integral Formula (or otherwise).

(b): On and inside the disc bounded by γ the function $e^{1/z}$ has a Laurent series expansion (plug $1/z$ into the exponential power series) which converges uniformly on any region $|z| > \epsilon$, for $\epsilon > 0$. Moreover $e^{1/z} = 1/z + h(z)$ where h has a primitive on that region. Thus

$$\int_\gamma e^{1/z}\, dz = \int_\gamma \frac{1}{z}\, dz + \int_\gamma h(z)\, dz = 2\pi i + 0.$$

Alternatively, some versions of the Residue Theorem cover this situation and it is just a matter of finding the residue at $z = 0$.

3. Find a Laurent series expansion valid in some bounded annulus centered at 0 that contains the point $z = 2$ for the following function (explain briefly how the inner and outer radii of the annulus are determined):

$$f(z) = \frac{z}{1 - z^2} + \frac{6}{(z-4)^2}.$$

Solution: As usual, write

$$f(z) = \frac{-z}{z^2} \frac{1}{1 - (1/z^2)} + \frac{1}{16} \frac{6}{(1 - (z/4))^2}$$

Since $1/(1-u)^2 = \frac{d}{du} 1/(1-u)$ we obtain from familiar geometric series expansions:

$$f(z) = \frac{-1}{z} \sum_{n=0}^{\infty} \frac{1}{z^{2n}} + \frac{3}{8} \sum_{n=1}^{\infty} n(z/4)^{n-1}.$$

The first series converges for $|z| > 1$ and the second for $|z| < 4$.

4. Use the calculus of residues to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1}\, dx.$$

Solution: For $R > 0$ let C_R be the semicircle in the upper half plane with diameter the interval $[-R, R]$, let γ_R be the closed curve consisting of C_R and $[-R, R]$ (oriented counterclockwise), and let $f(z) = e^{2iz}/(z^2 + 1)$. Since $\cos(2x) = \mathrm{Re}(e^{2ix})$, the integral may be found by taking the real part of $\int_{\gamma_R} f(z)\, dz$ as $R \to \infty$. This is because, by familiar estimates, the integral of f along C_R tends to 0 as $R \to \infty$. For all $R > 1$, $f(z)$ has one pole inside γ_R: a simple pole at $z = i$ with residue $e^{-2}/2i$. Thus the specified integral is $2\pi i(e^{-2}/2i) = \pi/e^2$ (which is already a real number).

5. Prove that if f is entire and there are positive real numbers A, B and k such that $|f(z)| \leq A + B|z^k|$ for all $z \in \mathbb{C}$, then f is a polynomial.

Solution: By a standard Cauchy estimate, the Nth derivative of f vanishes at 0 for all $N > k$. Thus the Taylor series for f expanded about the origin is a polynomial.

6. Let f be analytic on the closed unit disc $\overline{\mathbb{D}}$, and assume $|f(z)| < 1$ on its boundary. Prove that there is one and only one point $z_0 \in \mathbb{D}$ such that $f(z_0) = z_0$.

Solution: Apply Rouché's Theorem to the functions $f(z) - z$ and $-z$ to obtain this.

7. (a) Exhibit an entire function, $P(z)$, that has simple zeros at the numbers $\sqrt{n}$ for each positive integer n, and no other zeros.

(b) For the function P you gave in part (a), describe P'/P as an infinite series (not necessarily a Taylor series however).

Solution: Since $\sum_{n=1}^{\infty} (1/\sqrt{n})^3$ converges, the following Weierstrass product is entire:

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{\sqrt{n}}\right) \exp\left(\frac{z}{\sqrt{n}} + \frac{1}{2}\left(\frac{z}{\sqrt{n}}\right)^2\right)$$

Its logarithmic derivative is

$$\frac{P'}{P}(z) = \sum_{n=1}^{\infty} \frac{z^2}{n(z-\sqrt{n})}.$$

Alternatively, one may know that $1/\Gamma(z)$ is entire with simple zeroes at the non-positive integers. Thus $1/\Gamma(1-z^2)$ provides an example of the desired type. One then needs to know the product formula for $\Gamma(z)$, which leads to a similar infinite series.

8. Define $f(z) = \displaystyle\int_0^1 \frac{dt}{1+tz}$.

(a) Show by using Morera's Theorem that f is analytic on the open unit disc $\mathbb{D}$.

(b) Find a power series expansion for $f(z)$ valid on $\mathbb{D}$.

Solution: (a): Let γ be a closed curve in $\mathbb{D}$ (it is sufficient to consider γ just a triangle or rectangle). Since $1/(1+tz)$ is continuous on the compact set $\gamma \times [0,1]$ it is permissible to interchange order of integration:

$$\int_\gamma f(z)\,dz = \int_\gamma \int_0^1 \frac{dt}{1+tz}\,dz = \int_0^1 \int_\gamma \frac{1}{1+tz}\,dz\,dt.$$

For every $t > 0$ the integrand has a primitive $\frac{1}{t}\log_{pr}(1+tz)$ in the variable z on $\mathbb{D}$, and so each inner integral is zero by Cauchy (we can discard the single point $t = 0$ when computing the outer integral). By Morera f is analytic on $\mathbb{D}$.

(b): By using the familiar power series expansion for $1/(1+u)$ we obtain

$$\begin{aligned} f(z) = \int_0^1 \frac{dt}{1+tz} &= \int_0^1 \left(\sum_{n=0}^{\infty} (-1)^n (tz)^n\right) dt \\ &= \sum_{n=0}^{\infty} \left(\int_0^1 (-1)^n t^n\,dt\right) z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^n. \end{aligned}$$

9. Let $P(z)$ and $Q(z)$ be polynomials with degree $Q \geq$ degree $P + 2$. Prove that

$$\sum_{z_i} \mathrm{Res}_{z=z_i} \frac{P(z)}{Q(z)} = 0$$

where the sum is over all poles z_i in $\mathbb{C}$ of the rational function $\dfrac{P}{Q}$.

Solution: Let C_R be the circle of radius R centered at the origin. By cancelling common factors if necessary, we may assume P/Q is in lowest terms, so its poles occur at the zeros of Q. For any R greater then the largest modulus of a zero for Q, by the Residue Theorem

$$2\pi i \sum_{z_i} \mathrm{Res}_{z=z_i} \frac{P(z)}{Q(z)} = \int_{C_R} \frac{P(z)}{Q(z)}\, dz. \tag{9.1}$$

The length of C_R is $2\pi R$; and it follows easily from the hypothesis that the integrand is asymptotically bounded in modulus by a constant times $1/R^2$. Thus the usual estimates on the size of $\int_{C_R} P/Q\, dz$ show that this integral tends to zero as $R \to \infty$. Since the left hand side of (9.1) is constant, it must be 0, as needed.

10. Suppose f is analytic on the punctured unit disc $\mathbb{D} - \{0\}$ and the real part of f is positive there. Prove that f has a removable singularity at 0.

Solution: Consider $g = (1 - f)/(1 + f)$, which is bounded by 1 on the set where Re $f > 0$. The filled-in value of $g(0)$ is some b with $|b| \leq 1$. By the Maximum Principle (f isn't identically infinite, therefore $|g| < 1$ for z near 0), $|b| < 1$. The filled-in value of $f(0)$ is $g^{-1}(b)$, which is a finite complex number.

Alternatively, by Casorati-Weierstrass f does not have an essential singularity at 0. If f has a pole at 0, one can write $f = cz^{-n}(1 + h(z))$ where h is analytic and $h(0) = 0$. Then it is easy to see that the real part of f must be negative arbitrarily close to 0.

Or, one can consider $g(z) = e^{-f(z)}$ at zero (which is bounded and analytic on the punctured disc), taking special care to eliminate the case when $g(0) = 0$.