Combinatorics Qualifying Exam
January 2026

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions.

PhD Pass: Three numbered questions solved completely, or two solved completely with substantial progress on another two.

MS Pass: Substantial progress on three questions.

Question 1

Let $\chi(G)$ denote the chromatic number of a graph G.

(a) Show that $\chi(G) \geq n/\alpha(G)$, where n is the order of G and $\alpha(G)$ the independence number.

(b) Show that if $\chi(G) = k$ then G has a subgraph of minimum degree $k - 1$.

(c) Show that every graph G with m edges satisfies

$$\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}.$$

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Question 2

(a) State Hall's Theorem for matchings in bipartite graphs.

(b) Show that if a bipartite graph is regular, it satisfies Hall's Condition.

(c) Show that if M is matching in a graph G that is suboptimal (contains fewer edges than some other matching in G), then G contains an augmenting path with respect to M. Recall that an augmenting path is a path whose first and last edge are not in M, and alternates otherwise.

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Question 3

Let Par denote the set of all integer partitions. Given a partition $\mu \in \text{Par}$, let $|\mu|$ denote its size (i.e., the sum of its parts).

(a) Give an ordinary generating function for $\sum_{\mu \in \text{Par}} x^{|\mu|}$. Justify your formula.

(b) Let OddPar denote the set of integer partitions, all of whose parts are odd. Let DisPar denote the set of integer partitions, all of whose parts are distinct. Prove that

$$\sum_{\mu \in \text{OddPar}} x^{|\mu|} = \sum_{\mu \in \text{DisPar}} x^{|\mu|}.$$

(c) Give a bijection between OddPar and DisPar. *You may communicate your bijection using a (nontrivial) example. You do not need to prove that it's a bijection.*

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Question 4

(a) State the elementary Knuth relations (i.e., transformations)

(b) Row insert 3 into the semistandard Young tableau

1	2	3	4	4
2	5	5		
3	6			
5				

. Explain how the Knuth relations pertain to the row-insertion steps.

(c) Explain the rules for column insertion.

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Question 5

(a) Define the independent set polytope and the base polytope of a matroid M on ground set E.

(b) Show that the base polytope of M is a face of its independent set polytope.

(c) Give an example of a matroid M for which the base polytope is not simple (i.e., has a vertex contained in more than $\dim P$ facets).

(d) Give an example of a polytope which is a generalized permutahedron but is not the base polytope of any matroid.

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Question 6

Let M be a loopless matroid on ground set E.

(a) Define the lattice of flats $\mathcal{L}(M)$ of M. Give a characterization of lattices of flats in terms of lattice-theoretic properties.

(b) Describe explicitly the lattice of flats of the uniform matroid $U_{r,n}$.

(c) Define the fine Bergman fan $\mathcal{B}(M)$ of a matroid M. Recall that the fine Bergman fan is the version of this fan which is defined using chains of flats.

(d) Explain why the Bergman fan is a pure fan of dimension $\mathrm{rk}(M) - 1$.

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