Combinatorics Qualifying Exam
August 2025

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions, each with three parts (a,b,c).

PhD Pass: Three numbered questions solved completely, or two solved completely with substantial progress on another two.

MS Pass: Substantial progress on three questions.

Question 1

We define the union of two graphs as follows: $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

(a) Show that
$$\chi(G_1 \cup G_2) \leq \chi(G_1) \cdot \chi(G_2),$$
where $\chi(G)$ is the (vertex) chromatic number of G.

(b) Give an example of two graphs G_1 and G_2 with $\chi(G_1) = \chi(G_2) = 2$ and $\chi(G_1 \cup G_2) = 4$.

(c) Show that for any k there exist two graphs G_1 and G_2 such that $\chi(G_1) = \chi(G_2) = k$ and $\chi(G_1 \cup G_2) = k^2$.

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Question 2

Every part of this question is about graph vertex-connectivity (as opposed to edge-connectivity).

(a) State Menger's theorem.

(b) Give two (different) examples: one of a graph G and edge $e \in E(G)$ such that G is 2-connected and $G - e$ is not, and one of a graph G and edge $e \in E(G)$ such that G is 2-connected and G/e is not.

(c) Show that if a graph $G \neq K_3$ is 2-connected, then either $G - e$ or G/e is 2-connected.

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Question 3

(a) Let a_n denote the number of ways n identical balls can be placed in 2025 labeled bins. Using first principles, write down in closed form an expression for the ordinary generating function $A(z) = \sum_{n \geq 0} a_n z^n$.

(b) Modify your answer to part (a) to give a closed form expression for the ordinary generating function $B(z) = \sum_{n \geq 0} b_n z^n$ where $b_n = a_n$ if n is even and $b_n = 0$ if n is odd.

(c) Using the theory of generating functions, find a functional equation for the ordinary generating function $D(x)$ of $\{n^5\}_0^\infty$ in terms of the generating function $C(x) = (1-x)^{-1}$. (To clarify: $D(x) = 0^5 + 1^5 x^1 + 2^5 x^2 + \cdots$.)

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Question 4

(a) Let $\Lambda_n(N)$ denote the homogeneous component of degree n in the ring of symmetric functions in N variables. What is the dimension of $\Lambda_n(N)$? Explicitly write down *as polynomials* a basis of $\Lambda_3(3)$.

(b) Explain how any Schur function can be expanded in terms of the monomial symmetric functions. Use your expression to compute $s_{2,2}$.

(c) Compute the Schur function $s_{2,1,1}$ using a Jacobi-Trudi determinant.

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Question 5

Recall that a matroid has many equivalent axiom systems. This problem concerns the independent sets and rank function axiom systems.

(a) Give the independent sets definition of a matroid.

(b) Give the rank function definition of a matroid.

(c) Assume that a matroid M satisfies the independent sets axioms, but you do not know that it satisfies the rank function axioms. Use the independent sets to define the rank function, and then use the independent sets axioms to prove that this rank function satisfies the necessary properties (axioms) of a matroid rank function.

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Question 6

Recall that the fundamental theorem of polytope theory states that a polytope can be defined in two equivalent ways.

(a) Give the H-presentation, i.e. half-space, definition of a polytope.

(b) Give the V-presentation, i.e. vertex, definition of a polytope.

(c) Assume you do not know the fundamental theorem of polytope theory holds. Let P be a polytope given by some half-space presentation. Prove that P has some vertex presentation. Hint: First, produce a candidate set of vertices using the given inequalities and induction on dimension. Then take an arbitrary point x in P and take a line passing through x. Use this line, and again induction on dimension, to prove that x can be described in terms of the candidate vertices as desired.

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