Numerical Analysis Qualifying Exam
University of Vermont
January 2026

Letter assigned by proctor (not your name): _____

Instructions:

- You will have four hours to complete the exam.
- Please do not write your name(s) on the examination sheets and/or submitted exam. Instead, please write the "code letter" communicated to you by the Graduate Director. Please number the pages of your work.
- No phones, notes or books are allowed. You may bring a pocket calculator (to perform division, multiplication, exponentiation, etc.) of numbers. However, all other operations, e.g., matrix multiplication, must be performed by hand on paper.
- Seven (7) problems will be assigned, of which problems 1-4 will be based on the 5737 material and problems 5-7 will be based on the 6737 material.
- Make sure to provide explanations to all steps of your solutions and to explain all answers.

PhD passing criteria: Four (4) problems must be completed, and one (1) problem must be attempted. At least two (2) problems from # 1-4 (group 1) and at least two (2) problems from # 4-7 (group 2) must be completed. Note that Problem 4 can count towards either group, but not both. A problem is considered complete if the score for it is at least 85%. To have attempted a problem, you must correctly outline the main idea of the solution and begin the calculation, but need not finish it. In numeric terms, the score for an attempted problem must be at least 60%.

MS passing criteria: MS candidates will be evaluated on the 5737 material only (i.e., on Problems 1-4). They must complete two (2) problems and attempt (see above) one (1) additional problem.

1. (a) State the Secant method for solving the equation $f(x) = 0$ and explain how it approximates the Newton method.

   (b) Suppose that the graph of $f(x)$ is concave down near the root x_*, i.e., $f(x_*) = 0$ and $f''(x_*) < 0$. Make a sketch showing how the next iteration x_{i+1} is found from the previous iteration(s) for both (i) the Newton method and (ii) the Secant method. Based on the sketch, predict with a brief explanation which method will converge faster to x_*. In your sketch, show one iteration only.

2. Simpson's rule for numerical integration is shown below

$$\int_a^b f(x)dx = \frac{b-a}{6}\Big[f(a) + 4f(\frac{a+b}{2}) + f(b)\Big]$$

   with error bounded by the term

$$\frac{(b-a)^5}{90} f^{(iv)}(c_0)$$

   for some $c_0 \in [a, b]$. Show that cutting the interval $[a, b]$ in half decreases the integration error (approximately) by a factor of 16.

3. (a) Compute a QR-decomposition of the matrix $\mathbf{A}$ below. (b) Suggest a type of problem for which the decomposition would be useful and explain why.

$$\mathbf{A} = \begin{bmatrix} 4 & 8 & 1 \\ 0 & 2 & -2 \\ 3 & 6 & 7 \end{bmatrix}$$

4. (a) Use the method $w_{n+1} = w_n + \frac{h}{2} f(t_n, w_n) + \frac{h}{2} f\big(t_{n+1}, w_n + hf(t_n, w_n)\big)$ and a time step of size $h = \frac{1}{10}$, to approximate $y(\frac{1}{10})$ for the following system:

$$\begin{aligned} y' &= 2y - t \\ y(0) &= 1 \end{aligned}$$

   (b) Use stability analysis to determine if the backward Euler method with the above h will produce a solution without exponential growth.

Problem 5

Set up a second-order boundary-value problem (BVP) for the height of a ball thrown from the top of a 90-meter tall building with an unknown initial velocity and reaching the ground in 3 seconds. Assume that the motion occurs only in the vertical direction and the only force acting on the ball is gravity; use the approximate value $g = 10$ m/s^2 for the acceleration of gravity.

Solve the above BVP for the height of the ball by the shooting method. The latter must employ the simple Euler method with $h = 1$ second to solve each initial-value problem. (That is, find the height of the ball at 1 and 2 seconds.)

Problem 6

Consider the Wave equation

$$u_{tt} = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad t > 0$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad t > 0$$

and the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le L$$

(where $f(x)$ and $g(x)$ are prescribed functions).

(a) Write out an explicit finite difference scheme for the Wave equation with global/discretization error of the order $O(\tau^2 + h^2)$, where τ and h are the step sizes in time and space, respectively. Explain why this scheme has the required order. (You may refer to certain finite-difference discretizations having a certain order *relative to a certain grid point*, but you do *not* have to prove that they have this order.)
Use this scheme to give an explicit equation in *matrix form* to express the approximate solution at time level $n+1$ in terms of the solutions at time levels n and $n-1$.
(Hint: Handle the time derivative term as when discretizing the equation of the harmonic oscillator with second-order accuracy.)

(b) To start this scheme, one will need the solution at the first two time levels, $n = 0$ and $n = 1$. Use the two given initial conditions to find values of the solution at these two levels.
Importantly, make sure that the order of accuracy of your solution at $n = 1$ is consistent with the desired order of the scheme. *Explain* briefly why the order of the solution at $n = 1$ which you chose will indeed guarantee the order of the scheme to be $O(\tau^2 + h^2)$.
(Hint: You will need to use the Wave equation itself.)

Problem 7

Consider the Heat equation on the whole line:

$$u_t \;=\; u_{xx}, \qquad -\infty < x < \infty\,.$$

Use the von Neumann analysis to determine if there is any range of parameter

$$r \;=\; \frac{\tau}{h^2}$$

for which the following scheme for the Heat equation,

$$\frac{U_m^{n+1} - U_m^{n-1}}{\tau} \;=\; \frac{1}{4}\left(\frac{U_{m+1}^{n} - 2U_m^n + U_{m-1}^n}{h^2} \;+\; 3\,\frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2} \right),$$

would be stable.

If it can be stable, determine the stability range for r.

If it can never be stable, make sure to determine *which* Fourier harmonics of the numerical solution are unstable (i.e., you must determine *all* such unstable harmonics).