Numerical Analysis Qualifying Exam
University of Vermont
August 2025

Letter assigned by proctor (not your name): _____

➤ You will have four hours to complete the exam. Instructions and passing criteria: No calculators or other electronic gadgets allowed. Seven (7) problems will be assigned, of which problems 1-4 will be based on the 5737 material and problems 5-7 will be based on the 6737 material.

PhD passing criteria: Four (4) problems must be completed, and one (1) problem must be attempted. At least two (2) problems from # 1-4 (group 1) and at least two (2) problems from # 4-7 (group 2) must be completed. Note that Problem 4 can count towards either group, but not both. To have attempted a problem, you must correctly outline the main idea of the solution and begin the calculation, but need not finish it. MS passing criteria: MS candidates will be evaluated on the 5737 material only (i.e., on Problems 1-4). They must complete two (2) problems and attempt (see above) one (1) additional problem.

Make sure to provide explanations to all steps of your solutions and to explain all answers.

Section 1, 5737

1. Assume that you have a base-2 (binary) computer that uses a 4-bit normalized mantissa (1.xxxx), a 4-bit exponent, and a sign for each. Assume that all numbers are chopped, rather than rounded.

   a. Give both the normalized floating point representation *and* base-10 representation for machine epsilon *on this machine*.

   b. What real numbers have the same representation as 16.625_{10} *on this machine* if numbers are converted to binary, chopped, and then stored?

2. Compute the quadratic polynomial interpolating the data points $(-h, 0), (0, 1), (h, 0)$ using the Newton basis and integrate the result on the interval $[-h, h]$ using the Composite Trapezoid rule at the nodes $x_0 = -h, x_1 = -h/2, x_2 = 0, x_3 = h/2, x_4 = h$.

3. Compute a QR-decomposition of the matrix $\mathbf{A}$ below. Suggest a problem for which the decomposition would be useful and explain why.

$$\mathbf{A} = \begin{bmatrix} 4 & 8 & 1 \\ 0 & 2 & -2 \\ 3 & 6 & 7 \end{bmatrix}$$

Section 2, 6737

4. Method

$$Y_{n+1} = \frac{1}{3}(2Y_n + Y_{n-1}) + \frac{h}{3}(f_{n-1} + f_n + 2f_{n+1}),$$

where $f_n = f(x_n; Y_n)$, etc., can be used to solve an ODE $y' = f(x, y)$ or systems of first-order ODEs. Here and everywhere below, the prime stands for the full derivative with respect to x.

(a) Show that this method if second-order accurate.
Note 1: You do *not* need a formula for the Taylor expansion of a function of two variables to be able to solve this problem.
Note 2: In the process of solving this problem, you may need to use certain derivatives. You *must* indicate which variable your derivative is with respect to and *whether your derivative is partial or full*.

(b) Write out the difference equations that result when this method is applied to integrate the equation of the harmonic oscillator $u'' = -\Omega^2 u$.

5. Explain how you would solve the boundary-value problem

$y''' = p(x)y' + q(x)y + r(x)$, (note the third-order derivative on the left-hand side)

$y(0) = \alpha, \qquad y'(1) = \beta, \qquad y''(1) = \gamma$

by the shooting method. You should describe and explain all relevant details, but should *not* solve any equations.

6. (a) Propose a 2nd-order accurate discretization of the equation

$$(p(x)u_x)_x = q(x)u + r(x). \tag{1}$$

(b) Use this discretization to set up a linear system for the boundary-value problem given by Eq. (1) and by the boundary conditions

$$u(0) = a, \qquad u_x(1) = b; \tag{2}$$

where a, b are some given constants. Use $h = 1/3$ and write out each equation of the linear system in question. Make sure to use the 2nd-order accurate approximation for the Neumann boundary conditions at the right end point.

7. Consider the Heat equation

$$u_t \,=\, c\, u_{xx}, \qquad -\infty < x < \infty$$

where c =const> 0.

Use the von Neumann analysis to determine if there is any range of parameter

$$r \,=\, \frac{c\tau}{h^2}$$

for which the following scheme for the Heat equation:

$$\frac{U_m^{n+1} - U_m^{n-1}}{\tau} \,=\, c\, \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}\,,$$

would be stable.

If it can be stable, determine the stability range for r.

If it can never be stable, make sure to determine *which* Fourier harmonics of the numerical solution are unstable (i.e., you must determine *all* such unstable harmonics).