Differential Equations PhD Qualifying Exam

August 2024

Instructions:

- The examination contains two Sections, the first of which covers the material on initial-value problems for ODEs and the second, for boundary value problems and PDEs.
- To pass the exam, you must solve two (2) problems from each Section and attempt any two (2) additional problems. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations.
- Numeric criteria for passing: A problem is considered completed (attempted) if a grade for it is $\geq 85\%$ (≥ 60 %).
- Please do not write your name(s) on the examination sheets and/or submitted exam. Instead, please write the "code letter" communicated to you by the Graduate Director.
- You have four hours to complete the exam.

Problem 1

a. Convert the following 2nd-order IVP into a 1st-order system and solve explicitly.

$$\ddot{x} - 9\dot{x} - 10x = 0 \tag{1}$$

$$x(0) = 1 \tag{2}$$

$$\dot{x}(0) = 0 \tag{3}$$

b. Draw the phase portrait associated with (**??**) and determine the stability of the fixed point at the origin. Be sure to preserve proportions in his phase portrait.

Problem 2

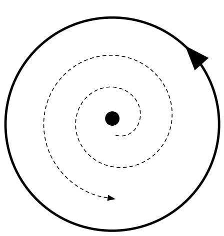

Figure 1: Example phase portrait of a 2-D system of ODE's having a trajectory $\Gamma(t)$ (dashed curve) with an ω-limit set consisting of a single limit cycle (solid curve). The fixed point in the figure is unstable (repelling).

The ω-limit set of a trajectory $\Gamma(t)$ is the set of points p such that there exists a sequence $t_n \to \infty$ with

$$\lim_{n \to \infty} \Gamma(t_n) = p \tag{4}$$

Assuming all orbits are bounded, sketch the phase portrait of a 2-D system of ODE's having a trajectory $\Gamma(t)$ with an ω-limit set consisting of

a. a single limit cycle and a single fixed point.

b. two limit cycles and a single fixed point.

c. two limit cycles and two fixed points.

Hint: a limit cycle may join two fixed points (heteroclinic), join a fixed point to itself (homoclinic), or contain no fixed points (as in Figure **??**).

Problem 3

Solve the following inhomogeneous IVP explicitly.

$$\dot{\mathbf{X}} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} t \\ 1 \end{pmatrix} \qquad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

Problem 4

Given the 1-D ODE

$$\dot{x} = \frac{\sin(x)}{x} - \mu \tag{6}$$

a. Classify the stability of all fixed points for $\mu = 0$.

b. Draw a bifurcation diagram for the ODE.

c. Find *all* values of μ for which there exactly 2 fixed points?

Problem 5

(a) Let $f(s)$ and $g(s)$ be square integrable on the infinite line, so that their Fourier transforms $\widehat{f}(\omega)$ and $\widehat{g}(\omega)$ exist, where

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx\,,$$

and similarly for $\widehat{g}$.
Prove the Parceval Theorem:

$$\int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx = \int_{-\infty}^{\infty} \widehat{f}^*(\omega)\, \widehat{g}(\omega)\, d\omega\,,$$

where the asterisk denotes complex conjugation.
(b) Compute the Fourier transforms of

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad g(x) = f(x-a), \quad \text{where} \quad a > 0\,.$$

(c) Use the result of parts (a) and (b) to compute

$$h(y) = \int_{-\infty}^{\infty} \left(\frac{\sin x}{x} \right)^2 e^{ixy} dx \, .$$

Problem 6

Put the boundary value problem (BVP)

$$u'' + a\, u' + \lambda u = 0, \tag{1a}$$

$$u(0) = 0, \qquad u'(1) = 0 \tag{1b}$$

into the Sturm–Liouville form. Here the prime denotes d/dx, a is an arbitrary constant, and λ is the eigenvalue of the BVP (1). All work asked for below must be done using the Sturm–Liouville form of (1). **No credit** will be given if you handle (1a) as an equation with constant coefficients.

(a) Show that two eigenfunctions $u_\lambda(x)$ and $u_\mu(x)$ corresponding to *different* eigenvalues λ and μ are orthogonal with a certain weight function. Make sure to provide all relevant details of this derivation, beginning with what this weight function must be and, further, including clear explanations of why certain terms in the computation of the inner product vanish.

(b) Consider a non-homogeneous BVP

$$v'' + 2v' + 3v = f(x), \tag{2a}$$

$$v(0) = 0, \qquad v'(1) = 0 \tag{2b}$$

Present its solution as a series expansion

$$v(x) = \sum_{n=1}^{\infty} f_n\, u_{\lambda_n}(x),$$

over the eigenfunctions $u_{\lambda_n}(x)$ of (1), where n is the sequential number of the eigenvalue λ_n. Your task is to determine, using the results of part (a), the expressions for the coefficients f_n.

Assume that λ_n and $u_{\lambda_n}(x)$ are known (but do **not** compute them).

Problem 7

Consider the BVP

$$\begin{aligned} &u_{xx} + u = a, \quad a = \text{const}, \\ &u(0) = 0, \quad u(\pi) = 1\,. \end{aligned} \tag{3}$$

(a) Find a (simple) change of variables from u to a new variable v that reduces BVP (3) to a BVP with homogeneous boundary conditions:

$$\begin{aligned} &v_{xx} + v = f(x), \\ &v(0) = 0, \quad v(\pi) = 0\,. \end{aligned} \tag{4}$$

Also, obtain the explicit form of $f(x)$.

(b) Find a formal series solution of BVP (4).

(c) What value(s) should the constant a in (3) have in order for the solution obtained in part (b) to exist?

Problem 8

Solve the following wave equation on the semi-infinite domain:

$$u_{tt} = u_{xx}, \qquad 0 \le x < \infty, \quad t \ge 0, \tag{5a}$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad x \ge 0, \tag{5b}$$

$$u_x(0,t) = h(t), \quad t \ge 0. \tag{5c}$$

Also, explain why functions f and h should satisfy a constraint

$$h(0) = f'(0), \tag{6}$$

where the prime denotes differentiation with respect to the argument.