Qualifying Exam on Differential Equations August 2019

INSTRUCTIONS: Two problems from each Section must be completed, and one additional problem from each Section must be attempted. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations, but do not need to finish them. Numeric criteria for passing: A problem is considered completed (attempted) if a grade for it is $\geq 85\%$ ($\geq 60\%$).

Time allowed: 3 hours

Section 1 (ODEs)

1. Sketch a phase portrait of the equation $x'' + x^2 - 1 = 0$. In doing so, you must present a derivation of all stationary points and their types.

2. Solve the following initial-value problem by any method:

$$x'' - 3x' + 2x = 10; \qquad x(0) = 0, \quad x'(0) = 0.$$

3. Consider a weakly nonlinear oscillator equation $x'' + x + \epsilon x^3 = 0$. For $\epsilon = 0$, its solution is $x|_{\epsilon=0} = A\cos(\omega t + \phi)$, where the amplitude A and phase ϕ are arbitrary constants (assumed to be $O(1)$) and frequency $\omega = 1$. Use the method of multiple scales to find how the frequency depends on the amplitude for $0 < \epsilon \ll 1$.
Note: Whenever you need a trigonometric identity for a power of sine or cosine, e.g., $\sin^2\theta = (1 - \cos 2\theta)/2$, you may obtain it by relating a trigonometric function to $\exp[\pm i\theta]$.

4. (1) Give the normal-form equations describing the:
(i) pitchfork bifurcation (*both* types) and
(ii) saddle–node bifurcation.
For each case, sketch the bifurcation diagram showing the behavior of the stationary solution as the function of the control parameter.

(2) (i) Sketch a diagram that shows what happens to the eigenvalues of the linearized system as the control parameter crosses the critical value when the original system undergoes a Hopf bifurcation.
(ii) Make sketches that illustrate how the phase portrait for the system differs for subcritical and supercritical Hopf bifurcation when the control parameter is near its critical value.

Section 2 (PDEs)

5. Solve the following wave equation on the semi-infinite domain:

$$u_{tt} = u_{xx}, \qquad 0 \le x < \infty, \quad t \ge 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \ge 0,$$
$$u_x(0,t) = 0, \quad t \ge 0.$$

6. For $0 \le x \le \pi$, solve the following heat equation with a source and initial-boundary conditions:

$$\phi_t = \phi_{xx} + w(x,t),$$
$$\phi(0,t) = 0, \quad \phi_x(\pi,t) = 0,$$
$$\phi(x,0) = f(x).$$

7. Use the method of characteristics to solve the problem:

$$\begin{aligned} u_t + uu_x &= 0, \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= e^{-x^2}, \quad -\infty < x < \infty, \end{aligned}$$

and express your solution in terms of the initial-condition function (perhaps implicitly). In particular, describe how you would evaluate your solution at, say, $x = 1$ and $t = 0.1$. In the (x,t)-plane, sketch several typical characteristics. When and where will shocks form?

8. Consider the following eigenvalue problem

$$\phi'' - x\phi + \lambda x^2 \phi = 0, \quad 0 \le x \le 1,$$

$$\phi(0) = 0, \quad \phi'(1) + \phi(1) = 0.$$

(1) Show that its eigenvalues are real;
(2) Show that eigenfunctions corresponding to different eigenvalues are orthogonal under a certain weighting function.