Combinatorics Qualifying Exam
January 2025

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions, each with three parts (a,b,c).

PhD Pass: Three numbered questions solved completely, or two solved completely with substantial progress on another two.

MS Pass: Substantial progress on three questions.

Question 1

Let G be a bipartite graph with bipartition $V(G) = A \cup B$.

(a) Suppose that $d(v) = k \geq 1$ for all $v \in V$ (G is k-regular). Show that $|A| = |B|$.

(b) Suppose that $d(v) = k \geq 1$ for all $v \in V$ (G is k-regular). Show that G has a perfect matching.

(c) Suppose that $\delta(G) \geq 1$ and that for every edge $ab \in E(G)$ with $a \in A$ and $b \in B$, we have $d(a) \geq d(b)$. Show that G has a matching that covers all vertices of A.

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Question 2

Part (b) is a generalization of (a). You do not need to answer (a) if you give a complete answer to (b). Part (c) can be answered without answering (a) and (b).

(a) Let G be an n-vertex graph. Suppose that $V(G)$ has a partition V_1, V_2, V_3 such that for every $1 \leq i < j \leq 3$ there is an $x \in V_i$ and $y \in V_j$ such that $xy \notin E(G)$. Show that

$$\chi(G) \leq n - 2,$$

where $\chi(G)$ is the chromatic number of G.

(b) Let G be an n-vertex graph. Suppose that $V(G)$ has a partition $V_1, \ldots, V_k$ such that for every $1 \leq i < j \leq k$ there is an $x \in V_i$ and $y \in V_j$ such that $xy \notin E(G)$. Show that

$$\chi(G) \leq n - k + 1.$$

Hint: Work by induction on k. Suppose that $V_1, \ldots, V_{k-1}$ have been colored using ***exactly*** *$n - |V_k| - k + 2$ colors. Then there is a set $\{y_1, \ldots, y_{k-1}\}$ such that $y_i \in V_i$ $1 \leq i \leq k-1$ and each y_i has a non-neighbor in V_k. Show that there is a color that appears* ***only*** *in the set $\{y_1, \ldots, y_{k-1}\}$, and proceed from there.*

(c) Show that

$$\chi(G) + \chi(\overline{G}) \leq n + 1,$$

where $\overline{G}$ is the graph complement of G. You may use the result from (b).

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Question 3

Let a_n be the number of ways to build a $2 \times 2 \times n$ pillar out of $2 \times 1 \times 1$ bricks. Similarly, define b_n to be the number of ways to build a $2 \times 2 \times n$ pillar with a $2 \times 1 \times 1$ notch cut out of the top face from $2 \times 1 \times 1$ bricks. (To clarify: for b_n, the height-n pillar takes the shape of a height-$(n-1)$ pillar with a $2 \times 1 \times 1$ brick on top, oriented so it is of height one.)

(a) Compute a_n and b_n for $n \in \{0, 1, 2\}$.

(b) Find recursions for a_n and b_n. Be sure to include initial conditions.

(c) Express the ordinary generating function $A(z) = \sum_{n\geq 0} a_n z^n$ as a rational function in z.

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Question 4

(a) Compute the Kostka number $K_{(3,3,2),(2,1,2,1,1,1)}$.

(b) Find the image of the permutation (in one-line notation) $[3, 5, 1, 6, 4, 8, 7, 2]$ under the Robinson-Schensted Correspondence.

(c) Let $\leq_{\text{lex}}$ denote the lexicographic order on partitions. For any partition ν, let ν' denote the tranpose partition. Prove or disprove: for all partitions $\mu, \nu \vdash k$, $\mu \leq_{\text{lex}} \nu$ if and only if $\nu' \leq_{\text{lex}} \mu'$.

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Question 5

(a) Define a uniform matroid U_n^r of rank r on n elements.

(b) Given a graph G, define the graphic matroid $M(G)$.

(c) Prove that U_n^r is graphic if $r \leq 1$ or $r \geq n - 1$.

(d) Prove that U_4^2 is not graphic.

(e) Classify when U_n^r is graphic. Hint: Observe that if U_n^r is graphic for some specific values of r and n, then any minor of U_n^r is graphic.

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Question 6

(a) Give the definition of a matroid in terms of its lattice of flats.

(b) Define a polytope and the face lattice of a polytope.

(c) Given an example of the face lattice of a polytope, which is not the lattice of flats of any matroid.

(d) Give an example of the lattice of flats of a matroid, which is not the face lattice of any polytope.

(e) Give an infinite family of posets, each of which is both the face lattice of a polytope and the lattice of flats of a matroid.

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