# COMBINATORICS QUALIFYING EXAM
January 2024

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions, each with three parts (a,b,c).

**PhD Pass:** Three numbered questions solved completely, or two solved completely with substantial progress on another two.

**MS Pass:** Substantial progress on three questions.

## Question 1

Define a tree to be a connected acyclic graph. Prove that the following statements are equivalent:

1. The graph $G$ is a tree.
2. For each $u, v \in V$ there exists a unique $u$-$v$ path in $G$.
3. The graph $G$ is edge minimally connected.
4. The graph $G$ is edge maximally acyclic.

You may use the fact that given two $u$-$v$ paths $P_1, P_2$ such that $P_1 \neq P_2$ the union $P_1 \cup P_2$ contains a cycle.

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## Question 2

1. Define an Eulerian tour of a graph. State and prove Euler's characterization of graphs with Eulerian tours.
2. Define the cycle space of a graph over $\mathbb{F}_2$. State and prove a characterization of the cycle space in terms of Eulerian subgraphs, i.e. subgraphs with an Eulerian tour.

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## Question 3

(a) The *Stirling number of the second kind* $S_{n,k}$ counts the number of partitions of an $n$-element set into $k$ non-empty blocks. So, for example, $S_{4,2} = 7$. Write down a recursion and initial conditions for $S_{n,k}$ and explain combinatorially why they hold.

(b) Use your recursion to find a functional equation for $A_n(y) = \sum_k S_{n,k} y^k$.

(c) Find a formula for the number of surjections from a $n$-element set to an $k$-element set that involves $S_{n,k}$.

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## Question 4

(a) State the Robinson-Schensted-Knuth (RSK) correspondence (i.e., what sets is it a bijection between).

(b) What image of the permutation 86741532 under RSK?

(c) Apply the correspondence to the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

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## Question 5

1. Define a uniform matroid $U_n^r$ of rank $r$ on $n$ elements.
2. Given a graph $G$, define the graphic matroid $M(G)$.
3. Prove that $U_n^r$ is graphic if $r \leq 1$ or $r \geq n - 1$.
4. Prove that $U_4^2$ is not graphic.
5. Classify when $U_n^r$ is graphic. Hint: Observe that if $U_n^r$ is graphic for some specific values of $r$ and $n$, then any minor of $U_n^r$ is graphic.

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## Question 6

1. Give the definition of a matroid in terms of its lattice of flats.
2. Define a polytope and the face lattice of a polytope.
3. Given an example of the face lattice of a polytope, which is not the lattice of flats of any matroid.
4. Give an example of the lattice of flats of a matroid, which is not the face lattice of any polytope.
5. Give an infinite family of posets, each of which is both the face lattice of a polytope and the lattice of flats of a matroid.

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