COMBINATORICS QUALIFYING EXAM
August 2022

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions, each with three parts (a,b,c).

PhD Pass: Three numbered questions solved completely, or two solved completely with substantial progress on another two.

MS Pass: Substantial progress on three questions.

Question 1

Graphs are assumed to be finite and simple.

(a) State Menger's Theorem regarding two equivalent definitions of k-vertex-connectivity of a graph G.

(b) Let G be a k-vertex-connected graph and e any edge $E(G)$. Show that the graph $G - e$, obtained by removing e from G, is $k - 1$-vertex-connected.

(c) A *chorded cycle* is a cycle C together with an edge that joins two non-consecutive vertices of C. Show that if G is 3-vertex-connected, then it must contain a chorded cycle as a subgraph.

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Question 2

Graphs are assumed to be finite and simple. For a graph G, let $\chi(G)$ denote the chromatic number and $\omega(G)$ the clique number.

(a) It is not difficult to see that $\chi(G) \geq \omega(G)$. Give an example of a graph G such that $\chi(G) > \omega(G)$.

(b) It is not difficult to see that if G is a graph on n vertices and $\chi(G) = n$, then we must have $\chi(G) = \omega(G)$. Show that if G is a graph on n vertices and $\chi(G) = n-1$, then we must have $\chi(G) = \omega(G)$. *Hint: consider the complement of G.*

(c) Show that if G is a graph on n vertices and $\chi(G) = n - 2$, it need not be true that $\chi(G) = \omega(G)$.

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Question 3

A *triangulation* of a convex n-gon, $n \geq 3$ is a selection of non-intersecting diagonals the n-gon such that its interior is partitioned into triangles. Let a_n be the number of such triangulations for a convex n-gon.

(a) Gather data by considering the triangulations of convex n-gon for $n \in \{3, 4, 5, 6\}$. Draw the triangulations in each case. What is a_n for each of these four values of n?

(b) Find a recurrence relation for a_n. Make sure to *explain* why your recurrence relation holds. You may use the convention $a_n = 0$ for $n \leq 1$ and $a_2 = 1$.

(c) Use generating functions to find a functional equation for $A(x) = \sum_{n\geq 0} a_n x^n$.

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Question 4

(a) State the Robinson-Schensted-Knuth (RSK) correspondence (i.e., what sets is it a bijection between).

(b) Apply the correspondence to the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

(c) How does the image of A^T (the transpose) relate to the image of A under RSK?

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Question 5

(a) Give the rank function axioms for a matroid.

(b) Give the flats axioms for a matroid.

(c) Describe the flats of a matroid M in terms of the rank function of M.

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Question 6

Let P be an n-dimensional polytope.

(a) Define the f-vector of P and write down the Euler-Poincaré relation for P.

(b) Write down the f-vector for the n-dimensional cube.

(c) Directly verify the Euler-Poincaré relation for the f-vector of the n-dimensional cube by application of the binomial theorem.

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