Algebra Qualifying Exam — August 2024

Please do not identify yourself on your work. The proctor will assign you a letter. Write this letter on the top right of each page of your work.

Format

- The exam contains three sections (Sections A, B, and C).
- Each section contains three problems.
- Each numbered problem has lettered subproblems (Parts (a), (b), (c), etc.).
- In case of mistakes, you should aim to solve more (sub)problems than the minimum required for the desired PhD/MS Pass.

Instructions

- You have four hours to complete this exam.
- When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass

- Four numbered problems solved completely. The set of problems solved completely must include one from each of Sections A, B, and C.
- Substantial progress on two additional problems.

MS Pass

- Nine lettered subproblems solved completely.
- You must solve at least one subproblem from each section.
- There must be two sections from which you solve at least three subproblems.
- Examples

Result	Section A	Section B	Section C
MS Pass	1abc	4abc	9abc
MS Pass	2a	4a, 6ab	7ab, 8abc
Fail	1abc, 2abc, 3abc		
Fail		4a, 5abc, 6ab	8abc
Fail	1abc, 3ab	4a, 5a	8ab

Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. Suppose G is a group of order $1225 = 5^2 \cdot 7^2$.

   (a) According to Sylow's Third Theorem, what are the possibilities for the number of Sylow 5-subgroups and the number of Sylow 7-subgroups?

   (b) Let H be a Sylow 5-subgroup and K a Sylow 7-subgroup. Prove that H, K, and G are all abelian.

   (c) List the isomorphism classes of groups of order 1225.

2. Suppose your friend has invented a machine that shuffles cards. It always rearranges a stack of cards in the same way relative to how they were inserted. Suppose you have thirteen cards arranged in order 1,2,3,4,5,6,7,8,9,10,11,12,13 that you insert into the machine. Then you take the cards in the order they are returned to you and stick them in again. The cards come out after this second shuffle in the order 10,9,12,8,13,3,4,1,5,11,6,2,7.

   (a) In what order did the cards emerge after the first shuffle?

   (b) Could your friend have designed a machine as above (i.e., a machine that acts by applying a fixed permutation) that would only return the original ordering of a stack of 13 cards after exactly 72 applications?

   (c) Writing σ for the permutation encoded by the machine from (a), explain what permutations are conjugate in S_{13} to σ. Give an example of such a permutation not equal to σ.

3. Let $G = \langle x, y | x^8 = y^2 = e, yxyx^3 = e \rangle$.

   (a) Show that G is isomorphic to a semidirect product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$.

   (b) Find the center of G and the order of xy.

   (c) Is G solvable?

Section B

4. (a) Prove a surjective ring homomorphism from a field onto a ring with more than one element must be an isomorphism.

   (b) Describe all ring homomorphisms (not necessarily unital) from $\mathbb{Z}/20\mathbb{Z}$ to $\mathbb{Z}/30\mathbb{Z}$.

   (c) Prove that the only (field) automorphism of the field of real numbers is the identity. *Hint: First prove that any such automorphism is strictly increasing and the identity on the rationals.*

5. Let R be a *Boolean ring*, that is, $a^2 = a$ for all $a \in R$.

   (a) Prove that R is commutative.

   (b) Prove that any Boolean ring with more than two elements has at least one zero divisor.

   (c) Prove that every prime ideal of R is maximal. *Hint: Let I be a prime ideal in R and consider the Boolean ring R/I.*

6. Let k be a field of characteristic not equal to 2. Let G be the two-element group $\langle g | g^2 = e \rangle$. Consider the group ring $A = k[G]$.

   (a) Prove that A, considered as a left A-module, is the internal direct sum of two simple left A-modules.

   (b) Show that every left A-module decomposes into a direct sum of simple left A-modules.

   (c) Now assume now that the characteristic of the field k is 2. Give an example of a left A-module that cannot be decomposed into a direct sum of two simple left A-modules.

Section C

7. Let ζ be a primitive 7-th root of unity in $\mathbb{C}$ and let $K = \mathbb{Q}(\zeta)$.

   (a) Describe the isomorphism type of the Galois group of $K/\mathbb{Q}$.

   (b) State the Fundamental Theorem of Galois Theory.

   (c) Draw the lattice of subfields of K. *Note: You do not need to provide generators for the subfields, but you should indicate their degrees.*

8. Let X be the set of 6×6 matrices A with entries from $\mathbb{Q}$ and characteristic polynomial $(x^4 - 1)(x^2 - 1)$.

   (a) Write down the lists of possible invariant factors $a_1(x), a_2(x), \ldots, a_m(x)$ where $a_m(x)$ is the minimal polynomial for A and $a_i(x) | a_{i+1}(x)$ for all i.

   (b) Write down the companion matrices for each of the invariant factors given above.

   (c) Find a representative of each similarity (i.e., conjugacy) class of X.

9. Fix an odd prime number p. Let $\alpha \notin \mathbb{F}_p$ lie in an extension field of $\mathbb{F}_p$ of degree r, and let $\beta \notin \mathbb{F}_p$ lie in an extension field of $\mathbb{F}_p$ of degree s. Assume r and s are distinct prime numbers.

   (a) Explain why $[\mathbb{F}_p[\alpha + \beta] : \mathbb{F}_p] \neq 1$.

   (b) Explain why $[\mathbb{F}_p[\alpha + \beta] : \mathbb{F}_p]$ is not r or s.

   (c) Prove that $\alpha + \beta$ lies in $\mathbb{F}_{p^{rs}}$ but not any smaller field.