Algebra Qualifying Exam — August 2022

PLEASE DO NOT IDENTIFY YOURSELF ON YOUR WORK. THE PROCTOR WILL ASSIGN YOU A LETTER. PLEASE WRITE THIS LETTER ON THE TOP RIGHT OF EACH PAGE OF YOUR WORK.

Format

The exam contains four sections (Sections A, B, B+, and C).

Each section contains three problems.

Section B+ contains new ring theory topics which we wish to include in future exams but are not part of the current syllabus. Exam takers are free to attempt these problems but can also ignore this section completely. Problems solved in section B+ will count towards your total of solved problems in section B.

Finally, each numbered problem has a certain number of lettered subproblems (Parts (a), (b), (c), etc.).

Instructions

You have four hours to complete this exam.

When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

PhD Pass:

Four numbered problems solved completely; the set of problems solved completely must include one from each of sections A, B (or B+), and C.

Substantial progress on two other problems.

MS Pass:

Nine lettered subproblems solved completely; the set of subproblems solved must include two subsets of three subproblems that are all from the same section (here B and B+ count as the same section).

The set of subproblems solved must include one from each section A, B (or B+), and C.

Section A

In this section you may quote without proof basic theorems and classifications from group theory as long as you state clearly what facts you are using.

1. In this problem, suppose that G is a group with $182 = 2 \cdot 7 \cdot 13$ elements.

   (a) What are the possibilities for $n_2(G)$ and $n_{13}(G)$, the number of Sylow 2-subgroups and 13-subgroups of G, respectively?

   (b) Suppose that H is a group with 14 elements. Prove that $H \cong C_{14}$ or $H \cong D_7$. (Hint: Consider the action of H by conjugation on its Sylow 7-subgroup.)

   (c) Prove that if G has order 182, then $G \cong C_{182}$, $G \cong D_{91}$, $G \cong C_{13} \times D_7$, or $G \cong C_7 \times D_{13}$. For each possibility, compute how many Sylow 2-subgroups G has, and how many Sylow 13-subgroups.

2. Let H be a subgroup of S_n such that $[S_n : H] = n$.

   (a) Consider the action of S_n acting by left multiplication on the right cosets of H, i.e. $\sigma \cdot \tau H = \sigma\tau H$. Prove that the associated permutation representation $\phi\colon S_n \to S_n$ is an isomorphism of groups.[1]

   (b) Show that the image of H under ϕ is isomorphic to S_{n-1}. Conclude that every subgroup H of S_n of index n is isomorphic to S_{n-1}.

3. Let G be a finite group and H be a proper subgroup of G. Prove that

$$G \neq \bigcup_{g \in G} gHg^{-1}.$$

   To be clear, you must show that for every finite group G and every proper subgroup H of G, $G \neq \bigcup_{g \in G} gHg^{-1}$.

[1] The action gives a group homorphism $S_n \to \mathrm{Perm}(S_n/H)$, where $\mathrm{Perm}(S_n/H)$ is the collection of bijections of sets from S_n/H to itself. Since $[S_n : H] = n$ we have $\mathrm{Perm}(S_n/H) \cong S_n$. The morphism $S_n \to S_n$ is called the associated permutation representation.

Section B

4. Let R be a ring with 1, and M be a left R-module. Recall that M is irreducible if $M \neq 0$ and 0 and M are the only R-submodules of M.

   (a) Prove that if $M \neq 0$, then M is irreducible if and only if M is cyclic and every nonzero element generates M.

   (b) Suppose that R is commutative, and that M is a non-zero R-module.[2] Show that M is irreducible if and only if M is isomorphic to R/I for some maximal ideal I of R.

5. (a) State the Fundamental Theorem of Finitely Generated Modules over a PID, carefully defining all terms.

   (b) Find all possible similarity classes of 3×3 matrices A over $\mathbb{C}$ such that $A^2 + 2A - 3 = 0$, i.e., give one matrix from each possible similarity class.

   (c) Give an example of two 4×4 matrices over $\mathbb{R}$ which have the same characteristic and minimal polynomials, but are not similar.

6. Let F be a field, and $M_n(F)$ be the vector space of $n \times n$ matrices over F.

   (a) Compute the dimension of $M_n(F)$ by exhibiting an explicit basis for this vector space.

   (b) Consider the subspace of $M_n(F)$ containing the matrices that have trace zero. Compute the dimension of this subspace by exhibiting an explicit basis for this subspace.

   (c) Let $f\colon M_n(F) \to F$ be a linear function such that $f(AB) = f(BA)$ for all matrices $A, B \in M_n(F)$. Prove that f is a scalar multiple of the trace function.

[2] Note that the assumption that R is commutative is not strictly necessary here.

Section B+

7. Let k be a field and consider the ideal $I = \langle x^2, xy + y^2 \rangle \subset k[x, y]$.

   (a) Show that the tuple $(x^2, xy+y^2)$ is not a Groebner basis for I in the lexicographic ordering with $x \succ y$.

   (b) Compute a Groebner basis for I in the lexicographic ordering $x \succ y$.

   (c) Find a reduced Groebner basis for I in the lexicographic ordering $x \succ y$.

8. Let K be a field.

   (a) Prove that the following are equivalent:

      i. Every maximal ideal in the polynomial ring $K[x_1, \ldots, x_n]$ has the form $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $a_i \in K$, $i = 1, \ldots, n$.

      ii. If I is a proper ideal in $K[x_1, \ldots, x_n]$, then there is some $a = (a_1, \ldots, a_n) \in K^n$ such that $f(a) = 0$ for all $f \in I$.

      iii. For every maximal ideal M in $K[x_1, \ldots, x_n]$, the natural inclusion map given by the composition $K \hookrightarrow K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]/M$ is an isomorphism.

   (b) Prove that if the statements in part (a) hold for K for some $n \geq 1$, then K is algebraically closed.

9. In this problem $\operatorname{Spec}(R)$ will denote the collection of prime ideals of a commutative ring R; it will be given the structure of a topological space by declaring the closed sets to be of the form $V(I) = \{P \in \operatorname{Spec}(R) \colon P \supset I\}$ for ideals $I \subset R$.

   (a) If I and J are ideals in R and P is a prime ideal such that $P \supset I \cap J$, show that $P \supset I$ or $P \supset J$.

   (b) If I and J are ideals in R, show that $V(I) \cup V(J) = V(I \cap J)$.

   (c) Let $S \subset R$ be a multiplicatively closed subset. Let $\ell_S \colon R \to S^{-1}R$ be the localization morphism given by $r \mapsto r/1$ for $r \in R$. If $I \subset R$ is an ideal show that
   $$\ell_S^{-1}(S^{-1}I) = (I : S)$$
   where $(I : S) = \{a \in R : \exists s \in S, sa \in I\}$.

   (d) Recall that a homeomorphism of topological spaces is an isomorphism in the category of topological spaces. Let $S \subset R$ be a multiplicatively closed subset. Show that $\operatorname{Spec}(S^{-1}R)$ is homeomorphic to $D(S) = \{P \in \operatorname{Spec}(R) \colon P \cap S = \varnothing\} \subset \operatorname{Spec}(R)$ given its subspace topology. You can do this by following these steps:

      i. Show that there is a bijection between S-saturated ideals of R and ideals of $S^{-1}R$. (An ideal I is S-saturated if $(I : S) = I$.)

ii. Show that if P is a prime ideal then P is S-saturated if and only if $S \cap P = \varnothing$. Hence every element $\operatorname{Spec} S^{-1}R$ has the form $S^{-1}P$ for some prime ideal P of R with $P \cap S = \varnothing$.

iii. Show that the map $\operatorname{Spec} S^{-1}R \to \operatorname{Spec} R$ given by $S^{-1}P \mapsto P$ is continuous with continuous inverse.

Section C

10. In this problem, let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 4$.

    (a) Compute the splitting field K and all of its subfields.

    (b) Compute the Galois group of K over $\mathbb{Q}$.

    (c) Prove that K is isomorphic to $\mathbb{Q}(\zeta_8)$, the eighth cyclotomic field.

11. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 3$. Let K be the splitting field of f over $\mathbb{Q}$ in $\mathbb{C}$. Suppose that $[K : \mathbb{Q}] = n!$.

    (a) Prove that f is irreducible over $\mathbb{Q}$.

    (b) If $\alpha \in \mathbb{C}$ is a root of f show that the only automorphism of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ is the identity.

    (c) If $n \geq 4$ show that $\alpha^n \notin \mathbb{Q}$.

12. Let $\mathbb{F}_2$ be the field with two elements and let $\overline{\mathbb{F}}_2$ be an algebraic closure of $\mathbb{F}_2$. For this problem, let $g(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$ and let $\alpha \in \overline{\mathbb{F}}_2$ satisfy $g(\alpha) = 0$. Finally, let $k = \mathbb{F}_2(\alpha)$.

    (a) Please give a complete list of all monic irreducible polynomials of degree 3 over $\mathbb{F}_2$.

    (b) How many elements β generate the field k over $\mathbb{F}_2$? To put this in other words, how many elements $\beta \in k$ are there such that $k = \mathbb{F}_2(\beta)$?

    (c) Explicitly determine all of the elements $\beta \in k$ that generate the field k over $\mathbb{F}_2$. Please write them in terms of α.

    (d) How many generators does the group $k^\times$ have?

    (e) Explicitly determine all of the generators of $k^\times$. Please write them in terms of α.