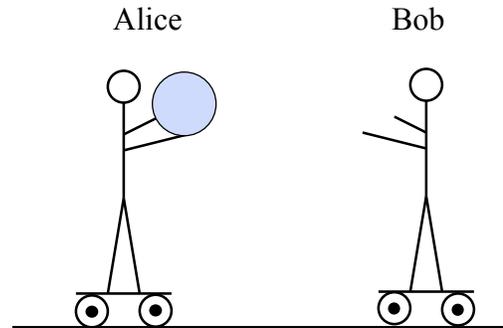


PART I. Do all four problems in Part I.

I.1 Classical Mechanics

Alice and Bob are on skateboards facing each other, initially at rest. Their respective masses, skateboard included, are m_A and m_B (see figure). Alice has a medicine ball of mass m , which she tosses to Bob. Bob catches the ball and then tosses it back to Alice who catches it. Assume that the speed of the ball when thrown is u relative to the thrower, either Alice or Bob. Also, neglect all dissipative forces like friction and air resistance.



- Find the velocity of each person after Alice catches the ball.
- Find the speed of each person in the limit that the ball is thrown back and forth (infinitely) many times. *Note:* You do not need the results from part (a) to do part (b).

I.2 Electricity and Magnetism

A certain amount of charge is placed on a thin conducting spherical shell of radius R . Inside the shell ($r < R$) is a spherically symmetric volume charge distribution $\rho(r)$. It is known that the electrostatic potential is

$$\Phi(r) = \begin{cases} \frac{\Phi_0 r^2}{R^2} & (0 \leq r \leq R) \\ \frac{\Phi_0 R}{r} & (R \leq r \leq \infty) \end{cases}$$

where Φ_0 is a constant.

- Find the electric field everywhere in space.
- Find the surface charge density at $r = R$.
- Find the volume charge density inside the shell.
- Use Gauss's Law to show that your answers in parts (b) and (c) are in agreement with your answer in part (a) for the electric field in the region $r > R$.

The following operators in spherical coordinates may be useful.

$$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

I.3 Quantum Mechanics

Consider the hydrogen atom, i.e. motion in a Coulomb potential $U(r) = -\alpha/r$.

Let us use the units $m = \hbar = \frac{1}{4\pi\epsilon_0} = 1$, and $\alpha = e^2$.

The ground state energy for $n = 1$, $\ell = 0$ ($n =$ principal quantum number, $\ell =$ angular momentum), is: $E_1 = -\alpha^2/2$.

- (a) Write down the Schrödinger equation for the radial part of the wave function $R_{10}(r)$, for the ground state. Show explicitly that in the ground state the wave function has the form $R_{10}(r) = Ae^{-\alpha r}$, where A is a normalization constant.
- (b) Find the normalization constant A , then calculate the expectation value $\langle r \rangle$ for the ground state.

Useful integral: $\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$, $n > 0$ and integer; $a > 0$.

I.4 Thermal Physics

- (a) Write down the partition function Z_1 for a single atom of mass m free to move in a cubical box of volume $V = L^3$. The energy values are given by

$$E_n = \frac{\pi^2 \hbar^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2),$$

where n_x, n_y , and n_z are any positive integers.

- (b) Show that Z_1 can be expressed as $Z_1 = n_Q V$, where n_Q is the quantum concentration,

$$n_Q = \left(\frac{mk_B T}{2\pi \hbar^2} \right)^{3/2}.$$

- (c) Show that $pV = k_B T$.

The following expressions may be useful:

$$\tau = k_B T$$

$$U = \tau^2 \frac{\partial \ln(Z_1)}{\partial \tau}$$

$$F = -\tau \ln(Z_1)$$

$$p = -\left(\frac{\partial F}{\partial V} \right)_\tau$$

$$\int_{-\infty}^{\infty} e^{-a(z-b)^2} dz = \sqrt{\frac{\pi}{a}}$$

PART II. Do only two of the four problems in Part II.

II.1 Experimental Physics

Three influential experiments in the development of physics were

1. The Michelson-Morley experiment
2. The Davisson-Germer experiment
3. The Stern-Gerlach experiment

Write a short paragraph on each experiment that should at the very least (a) describe what was done, (b) state the main findings and (c) indicate the role that the experiment played in our understanding of the physical world.

II.2 Mathematical Physics

The law of radioactive decay is written as $\frac{dN(t)}{dt} = -\lambda N(t)$, where $N(t)$ is the number of undecayed nuclei at time t and λ is the rate constant, a positive number.

Consider the following decay scheme: Start with N_0 nuclei of Isotope 1 at time $t = 0$. Isotope 1 decays to Isotope 2 with rate constant λ_1 ; Isotope 2 decays to Isotope 3 with rate constant λ_2 ; Isotope 3 is stable and does not undergo decay.

- (a) Write three equations giving the rates $\frac{dN_1(t)}{dt}$, $\frac{dN_2(t)}{dt}$ and $\frac{dN_3(t)}{dt}$. Verify that the sum of the three rates is zero.
- (b) Find $N_1(t)$, $N_2(t)$ and $N_3(t)$. Verify that the sum of the three functions at all times is N_0 .

II.3 Modern Physics

- (a) Starting from the basic definition of momentum $p = mv$, show that the relativistic momentum of a particle of rest mass m_0 can also be written as $p = m_0 c \sqrt{\gamma^2 - 1}$ where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$.
- (b) A neutral pion (π^0) of rest mass m_0 has total energy $E = 5 m_0 c^2$ in the lab frame. It decays into two photons that travel in exactly opposite directions. Find the energy of each photon.

II.4 General Physics

A uniform rod of length L and mass m is on the surface of a neutron star of radius R and mass M . The rod sticks radially out, i.e. one end is at R and the other end is at $L+R$. Consider the case where L is comparable to R .

- (a) Find the weight of the rod, i.e. the force of attraction between it and the neutron star.
- (b) Find the distance between the center of mass and the center of gravity of the rod.

PART A. Do all four problems in this part.

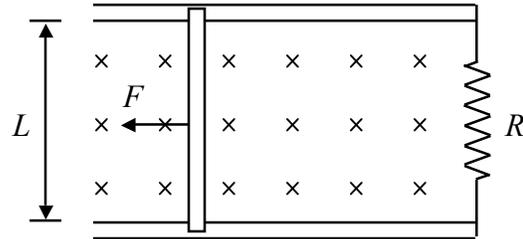
A.1 Classical Mechanics

A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and α ($\alpha > 0$) are constants. Let the potential energy be zero at the origin.

- Choose appropriate generalized coordinates and find the Lagrangian equations of motion.
- Show how angular momentum conservation about the origin can be obtained from the equations of motion.
- Show how total mechanical energy conservation can be obtained from the equations of motion.

A.2 Electricity and Magnetism

The figure shows two parallel rails of negligible resistance separated by distance L . One end of the rails is terminated with resistor R and a uniform magnetic field B is directed into the page. A rod of mass m is placed on the rails and pulled to the left with constant force F . Assume that the rail-rod contact is frictionless and that the force starts acting on the rod at $t = 0$.



- Find the terminal velocity of the rod v_T . Explain qualitatively why its dependence on the relevant parameters makes sense.
- Derive the equation of motion and simplify it using v_T as a known quantity.
- Find the velocity of the rod as a function of time. Check that it makes sense in the limiting cases.

A.3 Quantum Mechanics

A particle of mass μ is confined to a ring of radius r_0 . Suppose the particle is in the superposition state

$$|\psi\rangle = A(|0\rangle + 2|1\rangle + |-1\rangle + |2\rangle)$$

- (a) Find the normalization constant A .
- (b) What is the probability that a measurement of the energy will yield the value $E_1 = \frac{\hbar^2}{2\mu r_0^2}$?

A.4 Statistical & Thermal Physics

Consider a lattice of fixed hydrogen atoms; suppose that each atom can exist in four states as shown in the Table on the right.

State	Number of electrons	Energy
Ground	1	$-D/2$
Positive ion	0	$-d/2$
Negative ion	2	$d/2$
Excited	1	$D/2$

- (a) Find the thermal average number of electrons per atom as a function of the chemical potential μ , the temperature T , and the energies d and D .
- (b) Find the condition that the average number of electrons per atom be unity. The condition will involve the chemical potential μ and the energy d .

PART B. Do any two out of four problems in this part.

B.1 Modern Physics

Consider the hydrogen-like atom Li^{+2} in its ground state.

- (a) Find the rms radius $\sqrt{\langle r^2 \rangle}$ for the electron in this state.
- (b) Find the ionization energy for Li^{+2} .

B.2 Mathematical Physics

The temperatures at ends $x = 0$ and $x = L$ of a thin rod of length L with insulating sides are held at temperatures T_1 and T_2 until steady-state conditions prevail. Then, at the instant $t = 0$, the temperatures of the two ends are interchanged. The rod has uniform specific heat c_v and thermal conductivity κ . Find the resultant temperature distribution as a function of x and t ($t > 0$).

B.3 Contemporary Physics

In the model of a conductor that follows, electrons are approximated as free particles. An electron has charge $-e$ and effective mass m . The density of electrons is n per cubic meter. The background of the conductor is a fixed lattice with each site having charge $+e$ and the site density also n so that the conductor is neutral overall. Within a block of this conductor of rectangular area A and thickness d , consider a mode in which all electrons in the conductor move along the normal direction of the surface A by a displacement Δx . One surface of the block becomes positively charged because of the deficit of electrons while the other surface becomes negatively charged because of the excess electrons. Neglecting any fringe effect, the block now is a perfect parallel plate capacitor. This movement of electrons in the conductor is unsustainable because there is a restoring force due to the charge separation.

- (a) Find the net restoring force in terms of the parameters given above.
- (b) Consider all excess electrons as a single object of collective mass M . What is M ?
- (c) Derive the equation of motion of the above object of mass M and find the oscillation (resonance) frequency of this mode (this is the plasma frequency). You can leave your answer as the angular frequency but if you do, you must so indicate.

B.4 Experimental Physics

- (a) A ball of mass m and radius r moving in a liquid with velocity v experiences a viscous resistant force $f = 6 \pi r \eta v$, where η is the viscosity of the liquid. Describe an experiment that can be used to measure the viscosity of the liquid.
- (b) The total internal reflection technique has been used to generate an evanescent wave that propagates between a glass-water interface. The indices of refraction of glass and water are 1.5 and 1.3 respectively.
 - i. What is the critical angle of incidence that will allow total internal reflection when a light beam travels from glass to water?
 - ii. If the wavelength of the light is 630 nm and the angle of incidence is 70° , how does the light intensity decrease with the vertical distance from the glass-water interface?

PART A. Do all three problems in this part.

A.1 Classical Mechanics

A simple pendulum consists of mass m attached to a string of length L . Let φ be the angle measured from the equilibrium position.

- Write the Lagrangian $\mathcal{L}(\varphi, \dot{\varphi})$ and derive the equation of motion.
- Simplify (linearize) the equation of motion in the small angle approximation and solve it in the small angle approximation, $\varphi \ll 1$ subject to the initial conditions $\varphi(0) = \varphi_0$ and $\dot{\varphi}(0) = \omega_0$.
- Write down the total energy E of the pendulum. Is it conserved? Why or why not?
- Find the frequency ω of small oscillations.

A.2 Quantum Mechanics

Consider mass m at the end of a spring of constant k and frequency $\omega = [k/m]^{1/2}$.

- Write down the quantum mechanical Hamiltonian for this system in one dimension and give the energy eigenvalue corresponding to eigenstate ψ_n .
- Show that the Hamiltonian can be written in the alternative forms
 - $H = \hbar\omega(a_- a_+ - \frac{1}{2})$
 - $H = \hbar\omega(a_+ a_- + \frac{1}{2})$

where the ladder operators are defined by

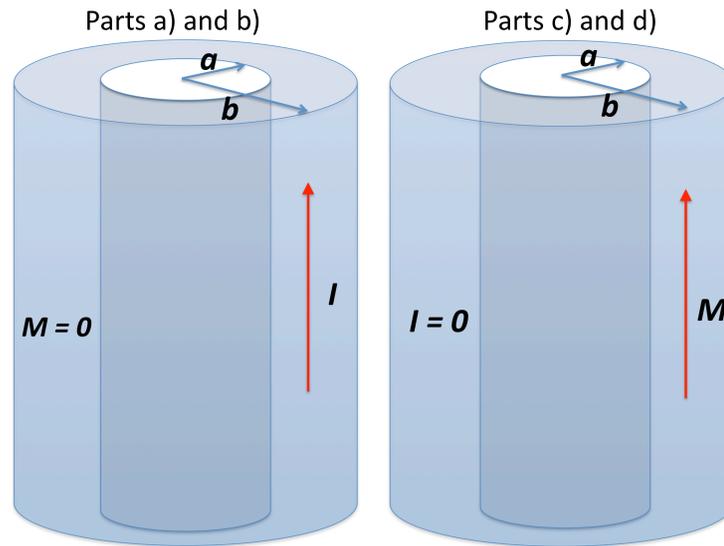
$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \text{ and } a_- = \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x)$$

- Find the commutator $[a_-, a_+]$.
- Show that
 - $a_+ a_- \psi_n = n \psi_n$
 - $a_- a_+ \psi_n = (n+1) \psi_n$

A.3 Electricity and Magnetism

A long non-magnetic cylindrical conductor with inner radius a and outer radius b carries total current I along its axis. The current density in the conductor is uniform.

- Find the current density vector \mathbf{J} everywhere in space.
- Find the magnetic field vector set up by this conductor as a function of radius inside the hollow space, within the conductor, and outside the conductor.
- Now assume that the same cylindrical conductor is made of a magnetic material and has uniform magnetization parallel to the conductor's axis (there is no longer any free current flowing through it). Find all the bound currents.
- Find the magnetic field setup by the conductor in part (c) as a function of radius inside the hollow space, within the conductor and outside the conductor.



Maxwell's equations in vacuum

$$\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Divergence Theorem: $\int_V \vec{\nabla} \cdot \vec{A} \, dV = \oint_S \vec{A} \cdot d\vec{S}$

Maxwell's equations in matter

$$\vec{\nabla} \cdot \vec{D} = \rho_{free}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$$

Curl Theorem: $\int_S \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \oint_C \vec{A} \cdot d\vec{\ell}$

PART B. Do one of the two problems in this part.

B.1 Statistical & Thermal Physics

Consider the Carnot cycle for an ideal monatomic gas ($C_V = 3/2$). The cycle is as follows: (i) Isothermal expansion while in contact with a reservoir at temperature T_H from volume V_1 to V_2 ; (ii) Isentropic expansion to volume V_3 ; (iii) Isothermal compression while in contact with a reservoir at temperature T_L to volume V_4 ; (iv) Isentropic compression back to V_1 . Assume that all processes are perfectly reversible.

- Sketch the cycle on a plot of pressure p vs. V with volume on the horizontal axis. Label points 1, 2, 3, and 4. What is the total change in entropy over one cycle?
- Show how the Sackur-Tetrode equation for a monatomic ideal gas, listed below, may be used to find the relationship between the ratios V_2/V_1 and V_3/V_4 .
- Find the heat input Q_H from the high temperature reservoir in terms of T_H , V_1 and V_2 . Also find the total work done by the gas, and show that the Carnot efficiency is obtained from W/Q_H .

Sackur-Tetrode equation:
$$S = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi mU}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right]$$

where:

- N = number of atoms
- k = Boltzmann constant
- V = volume
- U = internal energy
- h = Planck's constant

B.2 Mathematical Physics

- Write down the one-dimensional wave equation for transverse waves in a string and show that the vertical displacement of a point on the string can be represented by a traveling wave of the form $u(x,t) = f(x \pm vt)$ where v is the propagation velocity.
- A uniform string (mass density μ , tension τ and length ℓ) with fixed endpoints, $u(0, t) = u(\ell, t) = 0$ is plucked in the middle ($x = \ell/2$) giving rise to an initial displacement that is approximated by

$$u(x,0) = \begin{cases} \frac{2hx}{\ell}, & 0 \leq x \leq \frac{\ell}{2} \\ \frac{2h(\ell-x)}{\ell}, & \frac{\ell}{2} \leq x \leq \ell \end{cases}$$

Assume $h \ll \ell$. Find the initial energy of the string.

- Determine the Fourier amplitudes for the string.
- Find the energy in each normal mode and verify that energy is conserved.

Hint:
$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$