Math 255 - Spring 2022
Fermat's Little and Wilson's Theorems
15 points

This homework invites you to read about Fermat's (Little) Theorem and Wilson's Theorem in Section 6 of our textbook, and then to apply these theorems to answer the following questions.

1. Arrange the integers $\{2, 3, 4, \ldots, 21\}$ in pairs $a$ and $b$ that satisfy $ab \equiv 1 \pmod{23}$. In other words, find the inverse of each of these integers modulo 23.

2. Show that $18! \equiv -1 \pmod{437}$.

3. If $\gcd(a, 35) = 1$, show that $a^{12} \equiv 1 \pmod{35}$.
   Hint: Use Fermat's Little Theorem **and** the Chinese Remainder Theorem.

4. What is the last digit of $2022^{2022}$?

5. We have that

$$\begin{aligned} 6! &\equiv -1 \pmod{7} \\ 5!1! &\equiv 1 \pmod{7} \\ 4!2! &\equiv -1 \pmod{7} \\ 3!3! &\equiv 1 \pmod{7}. \end{aligned}$$

   (a) Perform the same sort of calculations modulo 11.

   (b) Guess a theorem from the data given to you and your data from part (a), and prove it.