Math 395 - Spring 2020
Homework 7

This homework is due on Monday, March 2.

All of these problems must be typed up.

1. Let R be a commutative ring with 1 which is a subring of the commutative ring S. Let P be a prime ideal of S.
   (a) Show that $P \cap R$ is a prime ideal of R.
   (b) Show that $P[x]$ is a prime ideal of $S[x]$.
   (c) Show that $P[x]$ is not a maximal ideal of $S[x]$.

2. Let R be a ring with 1 and let M be a *simple* left R-module (this means that M has no left R-submodules other than 0 and M).
   (a) If $\varphi\colon M \to M$ is a non-trivial R-module homomorphism (i.e. an endomorphism), show that φ is an isomorphism.
   (b) Show that if $m \in M$ with $m \neq 0$, then $M = Rm$.
   (c) Show that there is a left R-module isomorphism $M \cong R/\mathfrak{m}$ for some maximal left ideal $\mathfrak{m}$ of R.

3. Let R be a commutative ring with 1.
   (a) Prove that each nilpotent element of R lies in every prime ideal of R.
   (b) Assume that every nonzero element of R is either a unit or a nilpotent element. Prove that R has a unique prime ideal.

4. Classify all finitely generated R-modules, where R is the ring $\mathbb{Q}[x]/(x^2+1)^2$.

5. Let t be an indeterminate over $\mathbb{Q}$. Classify all finitely generated modules over the ring $\mathbb{Q}[t]/(t^9)$.

6. Let $R = \mathbb{C}[x, y]$ be the ring of polynomials in the variables x and y, so R may be considered as $\mathbb{C}$-valued functions on (affine) complex 2-space, $\mathbb{C}^2$, in the usual way (R is called the *coordinate ring* of this affine space). Let I be the ideal of all functions in R that vanish on both coordinate axes, i.e., that are zero on the set
   $$\{(a, 0) \mid a \in \mathbb{C}\} \cup \{(0, b) \mid b \in \mathbb{C}\}.$$
   (You may assume I is an ideal.)
   (a) Exhibit a set of generators for I. (Be sure to explain briefly why they generate I.)
   (b) Show that I is not a prime ideal.
   (c) Show that R/I has no nilpotent elements.